52. f. 7

A ROYAL ROAD TO GEOMETRY;

OR, AN

EASY AND FAMILIAR INTRODUCTION

TO THE

MATHEMATICS.

IN TWO PARTS.

I. PRACTICAL GEOMETRY, with Applications, and a familiar Introduction; for the uſe of Mechanics, &c.

Alſo, the Conſtruction of the Ellipſis; with ſome of its chief Properties demonſtrated.

II. ELEMENTS OF GEOMETRY ABRIDGED. Containing the whole Subſtance of Euclid's firſt ſix, the eleventh and twelfth Books; with ſeveral other, uſeful and valuable, Theorems; treated in the moſt brief, eaſy, and intelligent manner; for the uſe of Schools, &c.

Being an Attempt to render that moſt uſeful and neceſſary Science more univerſal, and practically applicable.

Interſperſed with Notes, critical, explanatory, and inſtructive.

By THOMAS MALTON.

To which is annexed, an Appendix, on the Theory of Menſuration of Superficies and Solids, as deduced from the Elements.

LONDON:

PRINTED FOR THE AUTHOR,

And ſold by Meſſ. ROBSON, Bond Street; DODSLEY, Pall Mall; BECKET, Strand; ROBINSON, Pater-noſter Row; RICHARDSON and URQUHART, Royal Exchange; TAYLOR, in Holborn; and by the AUTHOR, in Poland Street, Oxford Road.

MDCCLXXIV.

V

TO

SIR CHARLES FREDERICK;

Knight of the moſt honourable Order of the Bath, F. R. S. &c.

SIR,

AS there is an undiſputable Propriety in dedicating to a competent Judge of a Science, an Attempt to familiarize the Elements of it, I hope that propriety will plead an excuſe for this Preſumption; eſpecially, as the Subject is ſo nearly connected with the buſineſs of Your Perſonal Station, and too highly important to the Public, in general, to conceive, that any attempt at a clearer Inveſtigation can be beneath the notice of the Surveyor-General of his Majeſty's Ordnance, and one of the Repreſentatives of the People.

How far my Method of treating the Elements of Geometry may merit Your Approbation, I preſume not to imagine; farther than, as I flatter myſelf, the ſimplicity with which I have endeavoured to elucidate

the firſt Principles; to render the knowledge and the uſe of Geometry more general, and practically uſeful in common Life, where it is moſt wanted, may prove an additional inducement to your excuſing the liberty I have taken, in thus ſoliciting Your Patronage and Protection; for a Work, whoſe Subject, alone, is ſufficient to juſtify the attention You may pleaſe to honour it with; whatever may be thought of the manner in which I have treated it; having been, frequently (in order to comprize the whole Elements in leſs compaſs) under the neceſſity of deviating, conſiderably, from the Path of Euclid.

I am,

SIR,

with due reſpect,

Your moſt obedient

humble Servant

THOMAS MALTON.

ADVERTISEMENT.

IT may appear ſomewhat ſtrange, to call ſo copious a Work as this, an Abridgment; which, on examination, and comparing with thoſe who have tranſcribed Euclid, will be found to be a great abridgment of the Elements; both in the number of Theorems and length of the Demonſtrations.

But, they will find, in this Work, particularly in the 2nd, the 3rd, the 6th, 7th and 8th Books, ſeveral valuable Theorems which are not Euclid's; ſome of which are elementary, and really neceſſary to be known. Upon the whole, there are upwards of eighty other Theorems and Problems, excluſive of the Ellipſis.

In reſpect of the practical Part, with the Introduction, it may be deemed a compleat Work of itſelf; which, with the Appendix, is more than one third part of the whole.

The Applications, Notes and Remarks, neceſſary to a young Student, and the length of ſome (in the Fifth Book particularly) have ſwelled the Work conſiderably. The much greater number of Terms defined; various ways of performing the ſame Problem, and alſo of demonſtrating ſome Theorems; the number of Figures, and the Preamble to each Book, &c. alſo the manner of printing, in order to have the Figure always in View; ſo that, many Pages are not near full. All which have concurred to extend the Work, greatly beyond my firſt Deſign; yet, when compared with the cloſe printing of others, the Elements will be found abridged, almoſt half; nevertheleſs, it contains the whole in Subſtance. The full and perfect knowledge of all, and more than is contained in Euclid, may be acquired in a third part of the Time, and with infinitely more Eaſe, Pleaſure, and Satisfaction to the Learner (having no Aſſes Bridge to get over, but the Road ſmooth and even) conſequently, it may, with great Propriety, be called an Abridgment.

An INDEX,

Shewing where to find the Propoſitions of Euclid; in the Firſt, the Third, the Fifth and Sixth, the Eleventh and Twelfth Books; in caſes of Reference, to Euclid, by Authors in the Mathematics.

In the ſecond and fourth Books, the Propoſitions follow in the Order of Euclid.

N. B. The Problems, of Euclid, are collected together, in the firſt Part; amongſt other, ſelect Problems, in Practical Geometry.

Note. The Numbers in the Firſt Column are Euclid's; and, oppoſite to them, on the Right Hand, is ſhewn where each Propoſition may be found in this Work.

Book I.

Prop.	
1	Prob. — 11.
2	Prob. — 2.
3	Prob. — 3.
4	—— 8.
5	—— 9.
6	—— Cor. 3.
8	—— 7.
9	Prob. — 9.
10	Prob. — 8.
11	Prob. — 6.
12	Prob. — 7.
13	—— 1.
14	—— Cor. 1.
15	—— 2.
16 and 17	— 10.
18	—— 12.
19	—— Cor. 1.
20	—— 13.
21	—— 14.
22	Prob. — 14.
23	Prob. — 4.
24 and 25	are in Cor. to the 8th.
26	—— 11.
27	—— 4.
28	— Cor. 2.
29	— Cor. 1.
30	—— 5.
31	Prob. — 5.
32	—— 10.
33	Cor. to 15.
34	—— 15.
35, 36, & 37 & 38.	— 18.
39 and 40	— Cor.
41	—— 17.
42	Prob. — 20.
43	—— 19.
44	Prob. — 23.
45	Prob. — 22.
24, 20, & 21.	
46	Prob. — 17.
47	—— 20.
48	—— Cor.

Book III.

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1	Prob. — 39.
2	—— Ax. 5.
3	—— 1.
4	—— 2.
5 & 6	— Ax. 3.
7	—— 5.
8	—— 6.
9	— Cor. to 5.
10	—— Ax. 4.
11 and 12	— 7.
13	—— Ax. 6.
14	—— 3.
15	—— 4.
16	—— 8.
17	Prob. — 42.
18	— Cor. 3. 8.
19	— Cor. 2. 8.
20	—— 9.
21	—— 10.
22	—— 11.
25	Prob. 40.
26 and 27	— 10.
28 & 29	Cor. 2. of 3.
30	Prob. — 8.
31	—— 12.
32	—— 13.
33	Prob. 44.
34	Prob. 43.
35	—— 14.
36	—— 16.
37	Converſe.

Book V.

Prop.	
1	—— 1.
7	—— Ax. 4.
9	—— Ax. 5.
10	—— Ax. 6.
11	— Ax. 13.
12	—— 2.
14	— Ax. 12.
15	— Ax 7 & 8.
16	—— 4.
17	—— 7.
18	—— 6.
19	—— 3.
22	—— 9.
23	—— 10.
24	— Cor. to 3.
25	—— 11.

Book VI.

Prop.	
1	—— 1.
2	—— 2.
3	—— 3.
4	—— 4.
5	—— Cor. 1.
6	—— 5.
8	—— 7.
9	Prob. 3.

Prop.
10 Prob. 36.
11 Prob. 31.
12 Prob. 32.
13 Prob. 30.
14 and 15 — 8.
16 —— 9.
17 —— Cor.
18 Prob. 16.
19 —— 12.
20 —— 13.
21 —— Ax.
22 —— 15.
23 —— 11.
24 —— 17.
25 Prob. 38.
26 — Cor. to 17.
27 —— 18.
31 —— 16.
33 —— 19.

Book VII.

Prop.
1 —— Ax. 1.
2 —— Ax. 4.
3 —— 1.
4 —— 2.
5 —— Cor.
6 —— 3.
7 —— Ax. 3.
8 — Cor. to 3.
9 —— 4.
10 —— 5.
11 Prob. — 1.
12 Prob. — 2.
13 —— Cor.
14 —— 6.
15 —— 7.
16 —— 8.
17 —— 10.

Prop.
18 — — 9.
19 — Cor. 1.
20 and 22 — 13.
21 —— 12.
23 see NB. 2nd.
24 —— 14.
25 —— 15.
26, 27 } see NB.
28 —— 16
29, 30 and 31 are in the 17.
32 —— Cor. 1.
33 —— 18.
34 —— 19.
35 —— 22.
36 —— 23.
37 —— 24.
38 — Cor. 2. 9.

Book VIII.

Prop.
1 —— 14. 6.
2 — Cor. 1. D°.
5 and 6 — 6.
7 —— 3.
8 —— 7.
9 —— 8.
10 —— 5.
11 Cor. 1 & 2. 6.
12 Cor. 1 & 2. 7.
14 Cor. 3. 6.
15 — Cor. to 8.
18 — Cor. to 9.

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ERRATA.

Notwithstanding all my care and vigilance in revising the Press, some errors have escaped me; a clear proof of fallibility. If the following are corrected, I am in hopes that few more (literary errors excepted) will be found in the Work.

Practical Geometry.

Page. Line.

23.— 9. Art. 4. *for*, Circles, *read*, Circumferences.
30.— 1. Prob. *read*, To draw a Right Line.
38.— 8. *for*, CDAB, *read*, CDbB.
41.—10. *for*, intersecting F, *read*, intersecting *at* F.
47.— 2. N. B. *for*, CDG, *read*, CD *G*
67.— 4. *read*, to *the* fourth
75.— 3. Par. 2. *for*, AC, *read*, aC
98.—11. *for*, Def. 15. and 16. 7th. *read*, 1 and 6. 8th.
104.— 7. Prob. 7. *for*, H, *read*, K; and, *for*, K, *read*, H
107.—16. *for*, and, *read*, i.e. that is

In the Elements.

135.— 5. Theo. 2. *for*, BAG, *read*, EAG
147.—13. *for*, Hyp. AB, *read*, AC
156.— 9. Th. 2. *for*, AE, *read*, AD.
—14. *for*, Ax. 3. *read*, 2.
172.— 5. Bottom. *for*, 13th, *read*, 14
182.—13. *for*, Def. 42 *read*, 43.
183.— 7. Cor. 2. *for*, in A, *read*, in H
190.—10. Theo. 8. *for*, D in DG, *read*, DG in D.
195.—11. *for*, cross each, *read*, cross each *other*.

Page. Line.

223.— 5. B. *for* AF □ = FB □, *read*, AF □ = AE □ + EF □
225.— 8. Cor. *for*, Decagon, *read*, Duodecagon
240.—7. B. *for* equality of Ratios, *read*, Ratio of Equality
244.—2 and 4, Bottom. *for*, Ratio, *read*, Increase
257.—11. *for*, Def. 6. *read*, Ax. 6.
—19. *for*, A, B, *read*, *if* A : B
258.— 7. Post. *for*, Axiom, *read*, Postulate.
261.—10. Theo. *for*, to a, as A to B, *read*, to *a*, as B to A.
268.— 2. DEM. *for*, A and B, *read*, A and C
329.—10. *for*, 8. 4. *read*, 9. 4.
347.—10. DEM. *read*, Draw other Right Lines
349.— 3. DEM. *for*, on, *read*, *or* the other.
358.— 3. DEM. *for*, BK, *read*, BL
365.— 6. *for*, Ax. 4. 5. *read*, 5. 7.
391.— 3. *for*, Ax. 11. 7. *read*, Def.
408.—13. DEM. *read*, and having equal Altitudes

In the Appendix.

19.—5 & 6. Par. 4. *for*, K, *read*, F
25.—1, 3 & 4, Par. 3. *for*, G, *read*, E
32.— 6. *for*, Chord of 90, *read*, 60.

PREFACE.

THE mathematical World will, I doubt not, be ſurprized at a freſh publication of the Elements of Geometry, by one entirely unknown; and, on a plan very different from that of others who have wrote on the Subject. I hope they will ſuſpend their opinion, and not paſs a too haſty cenſure, on account of the obſcurity of the Author, till after they have given it a fair and candid peruſal, and then proceed to judgment with candour and impartiality.

I do not pretend to much knowledge in the Mathematics, having been brought up in a way of life, very different from my inclination; yet, what time I could ſpare from buſineſs and the demands of my family, I choſe to employ in ſuch ſtudies; and have, by dint of ſtudy, only, and without any other inſtruction, made ſome progreſs in mathematical Sciences; of which, Geometry is the firſt, and a ſure key to the reſt.

Since I have made myſelf, by ſelf-application, a Proficient in Geometry, and have made ſome branches of the Mathematics my Study and Profeſſion, I have often been ſurprized at the negligence and deficiency of our common Schools, for the cultivation of Youth who are intended to fill the middle ſphere of Life, in mechanic Trades, &c. They, almoſt in general, purſue one common Plan or track of Learning. After the firſt and neceſſary branches, Reading, Writing, and Arithmetic; which, indeed, might be acquired in half the time it uſually is; the next ſtep (if the Pupil has made a progreſs thro' Arithmetic in any reaſonable time) is the Grammar of the Latin Tongue through which, he ſweats and labours to little purpoſe. If the Pupil has three or four years to ſpare, before he goes out to buſineſs, he perhaps gets into the Cordery or Eraſmus; or, if he reaches Cornelius Nepos, he is looked on as a prodigy.

Now, it may reaſonably be aſked, for what purpoſe all this Time has been ſpent? which might have been employed to much

better purpoſe. For, what has mechanic Trades to do with Latin? any more than a common Porter or Carman with Logic; it may indeed complete him a Pedant or Coxcomb, but can never be of real uſe in his Profeſſion; even ſuppoſe he had made a tolerable proficiency, it could anſwer no purpoſe but to ſet him above his Employment, without being of any ſervice in it.

On the contrary (ſuppoſing no particular avocation is intended for the Student) if, inſtead of Latin, Geometry and Menſuration, &c. were introduced in all public, common Schools, I would aſk any perſon, who has conſidered theſe things, and their uſes in Life, which is moſt likely to turn to the Pupil's advantage? Is there a mechanic Profeſſion in which Geometry or Menſuration may not be of ſome uſe? in ſome particular ones it is well known to be of the greateſt, the foundation of it; and yet, altho' the Youth was particularly intended for that Profeſſion, it was, perhaps, never once ſo much as thought on; until, by too late experience, he finds the want of it: I mean all ſuch Trades as relate particularly to Building, in general. Had ſome Builders, whom I have known, been converſant in the Mathematics, or only in plane Geometry; inſtead of plodding on in a low ſphere of Employment, they would, if their natural, mechanical genius had been properly cultivated, have filled a more elevated ſtation.

For the uſe of ſuch, I have been at the trouble of compoſing this Treatiſe. If only the practical part is well inculcated, it will be of more ſervice, in common Life, than a proficiency in Latin can poſſibly be. If the young Pupil has a genius, and diſcovers a reliſh for mathematical Science, let him go on with the Elements; and if he acquires a competent ſhare of knowledge therein, it will then be time to conſider, what particular Profeſſion he either wiſhes or is deſtined for. In chooſing of which, regard ought particularly to be had to the Boys genius and diſpoſition, which will, ere this, be diſcernable. But, at all adventures, inſtead of flogging and driving a uſeleſs dead Language into a ſtupid Boy, which only renders him more ſo, let the practice of Geometry be introduced in its ſtead, in every common School; there is ſomething entertaining

to

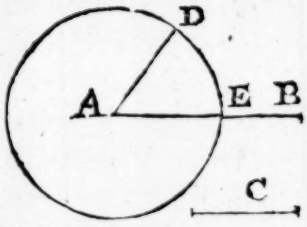
to the Mind, more than in burdening the memory with *As in præſenti*, and other rules of the Latin Grammar.

Accuſtoming Boys, early, to handle the Compaſſes and other drawing utenſils, in delineating all the Diagrams, as they proceed, will be an entertainment to them, and of great utility, rather than a perplexing ſtudy, and gradually enure them to Demonſtration; which, under a proper Tutor, they would ſoon have a reliſh for, and then they would proceed with pleaſure: beſides, it is an introduction to Drawing. A foundation being once laid in Geometry, they are then qualified to purſue any other branch of the Mathematics, ſuitable to the Profeſſion they are intended for; ſuch as Menſuration, Trigonometry, Navigation, Gunnery, Fortification, Architecture, naval or domeſtic, Surveying, &c. In ſhort, all the uſeful and neceſſary Employments, in the mechanic Arts, have their foundation in this moſt neceſſary Science; which, being acquired, will, moſt probably, make its Poſſeſſor ſtrike out of the common and vulgar track, and make him eminent and diſtinguiſhable, in whatever Profeſſion he is caſually fixed in; as he will have laid a ſolid and permanent foundation, in Theory; whereon, may, very probably, be erected, a laſting monument to his future Fame.

I have peruſed ſeveral Authors on the Subject, and find, that ſome have treated it in a manner ſcarce intelligible to a beginner, unleſs he has ſome knowledge of Algebra; others would be better underſtood and approved, if they did not dwell too much on ſelf-evident Propoſitions, which are, in themſelves, perfect Axioms. Perhaps, I ſhall be blamed for cenſuring, as uſeleſs, ſeveral Propoſitions in that famous Geometrician, Euclid; but muſt own, I cannot conceive of what uſe is all that tedious round about method, in the 2nd Problem, Book 1. *viz.* "To "put a right Line, at a given point, equal to a given Line," unleſs ſome particular direction, in reſpect of the other Line, was alſo given.

In Problem 3d. where the 2nd is applied, I aſk, for what uſe? and why, after having taken the line C in the Compaſſes, as Radius,

we may not as well cut off from the given Line (AB) at once, the meaſure of C, (at E) without firſt placing it from A to D? (See the Figure) nor can I perceive that the Demonſtration is, at all, the clearer for it.



As the knowledge I have acquired, in ſuch ſtudies, has been entirely from Books, without any other aſſiſtance; I may, perhaps, have been able to ſee deficiences or ſuperfluities in them, better than thoſe who have ſtudied under a Tutor; and, I dare venture to affirm, that very few, who are not already tolerably verſed in Geometry, will be able to form any Idea of what the 7th Propoſition, Book I. attempts to prove; who, from inſpection only of a proper Figure, might be fully convinced of the truth of it.

I have, in this Treatiſe, endeavoured to render every Propoſition eaſy and intelligible, to any capacity; and have omitted none that are uſeful, or neceſſary to demonſtrate the reſt; and, I will be bold to affirm, that thoſe who cannot, from it, acquire a knowledge of all that is requiſite, in Plane Geometry, will never be able to attain it at all. Where there is not a capacity to underſtand, they had better deſiſt from the undertaking. An equal talent is not given to all; and, for ſuch as have not a talent and a natural propenſity, to expect to attain any tolerable ſhare of knowledge, in Geometry, is aiming at impoſſibilities.

The 16th and 17th Propoſitions, Book 1ſt, are entirely uſeleſs; for ſince, in the 32nd, the external Angle is proved to be equal to the two remote Angles of the Triangle; and, that all the three Angles, of every Triangle, are equal to two right ones; it ſeems abſurd to prove, before hand, that any two of the Angles are leſs than two right Angles, and, that the external Angle, is greater than either of the remote ones. As if one ſhould undertake, firſt, to prove that five is greater than either two, or three, and afterwards, that it is equal to them both. I have, therefore, made free, to alter, in ſome meaſure, the Elements of the firſt Book; that, by a different arrangement, in tranſpoſing the places

of

of ſome Propoſitions, others, which dépend on them, may with more eaſe and elegance be demonſtrated.

I ſhall ever be of opinion with Tacquet, and ſome others, that, to attempt a formal Demonſtration of Propoſitions which are ſelf-evident, is involving a thing, in itſelf clear and conſpicuous, in darkneſs and obſcurity. I have always found more difficulty in demonſtrating, to another Perſon, ſelf-evident Propoſitions, than the moſt intricate of others; and, when done, have only confuſed the Idea which the Pupil had more perfectly from inſpection of the Figure. If a knowledge of ſeveral properties of Figures be acquired, which is neceſſary to elucidate other more ſublime Propoſitions, is not the eaſieſt method the moſt eligible? certainly it is; and, barely to be told ſeveral properties of Triangles is ſufficient for attaining the reſt. Therefore, in this Work, I have reduced ſome Propoſitions into Corollaries; as they are but a certain conſequence of the preceding Propoſition, or of ſome other.

Dr. Keill, in his Preface to his Tranſlation of Commandine, and alſo Mr. Cunn ſeems to think it an unpardonable fault in Tacquet, to omit the Demonſtrations of the 5th Book; and aſſerts, that not one Demonſtration, in the 6th, 11th and 12th, can be obtained without it. But, I muſt beg his pardon, for differing from him in opinion, and am, myſelf, a living witneſs againſt ſuch his aſſertion; for, I do aver, that, without any other demonſtration of the 5th Book, than what Tacquet has delivered, I have been able to go through all the other; and, without vanity, I think I underſtand them; but, of that, let this Treatiſe bear witneſs. I never could, at firſt, have had patience to go through the dry and tedious Demonſtrations, delivered in 25 Propoſitions, of the 5th Book of his Euclid; and, had it firſt fallen in my way, I ſhould certainly have lain it aſide before I had got through a fourth part of it; yet, I muſt acknowledge, that Tacquet is as much too brief as the other is tedious.

I cannot think it abſolutely neceſſary, in order to obtain a competent knowledge in Geometry, and of Proportion in particular, to tread exactly in the ſame ſteps with Euclid; therefore, I have made

made free to deviate, where I think it for the better, a readier or clearer way of inveſtigating. I have, likewiſe, ſeparated the Problems from the Theorems; making, of them, a diſtinct and ſeparate Book, for the uſe of Schools and for Mechanics in general; the full Demonſtration, of which, may be obtained from the Elements, to which I refer where it is neceſſary. I have given it the firſt place, contrary to ſome others who have made it the laſt Book, or otherwiſe diſpoſed of it. By which means, we are frequently at a loſs in the References, and are told to form Conſtructions, before we have learned how.

My reaſon for ſeparating Practical Geometry from the Theorems is, that there are numbers of People, to whom it may be of great uſe, who either have not leiſure or inclination to go through, or, perhaps, a capacity to underſtand the Demonſtrations of the previous Propoſitions, which diſcourages them from proceeding. But, ſuch as are not inclined to take it on credit, will readily obtain the Demonſtration from the Theorems.

To make the Work more compleat I have added, after Practical Geometry, a brief Theory of the nature and conſtruction of an Ellipſis, that moſt uſeful and valuable Figure. And alſo, as an Appendix to the whole, I have given a conciſe Theory of menſuration of Superficies and Solids; ſhewing their immediate and abſolute dependence on Geometry.

I have for ſome time debated, with myſelf, whether I ſhould publiſh a tract of Geometry or not; from which I have been deterred, through the perſuaſions of ſome particular Acquaintance; alledging, that there are more learned Treatiſes already publiſhed, than any I could produce: the truth of which, I readily aſſent to; but, I don't find that they are more intelligible for that, and have, therefore, determined to publiſh. My chief reaſon, for which, is to bring the whole Elements into leſs compaſs, and to abridge it of that tedious prolixity, which is in many Authors, on that Subject; in dwelling too long on ſuch Propoſitions as are clear and evident of themſelves; and, by that means, to render the ſtudy of Geometry more pleaſant and entertaining. 'Tis enough to deter any one from the purſuit of a Science, who find, in the Rudiments of it,

ſo

ſo much perplexity; for, I am perſuaded, that, unleſs a Perſon has a ſtrong inclination to it, and a good natural Capacity, 'tis not a very pleaſing Study, at the firſt, until they begin to feel the ſublime Truths it contains; and am therefore of opinion, that, the eaſier it is made in the begining, the more entertaining to young Students.

Can there need Demonſtration, that any two ſides of a Triangle, are greater than the third? is there a Perſon ſo ignorant as not to know it? it is implanted in us by Nature; every common Porter knows it, or practices it every Day. Who ever ſaw one of them traverſe two ſides of a Square, when he could croſs the Diagonal? and why is it? but, becauſe he knows it to be ſhorter than the two Sides. Is it not obvious, that the greateſt Angle of every Triangle, muſt be oppoſite to the greateſt Side? and can there be any need to demonſtrate the converſe? that the greater Side ſubtends, or is oppoſite to the greater Angle; is not the one contained in the other? it is trifling, to no purpoſe; for, all converſe Propoſitions may be Corollaries to the former. If two Sides of a Triangle, equal to two Sides of another Triangle, contain a greater or leſs Angle, the Baſe will be greater or leſs? Are not all theſe, and ſeveral more, obvious and clear, from a bare inſpection of the Figure? nay even without it; 'tis enough to be told they have ſuch properties, and not to loſe time in trying the patience of the Student, with a tedious and puzling Demonſtration of what he ſaw clearer before; for, if the thing is ſeen or known, what needs there more? is it intended to perplex, only, where it can be of no uſe? to diſguſt the beginner, before he is able to ſee any of the Beauties it contains? Yet, I do not omit theſe entirely, becauſe the whole Elements depend on them; but have endeavoured to treat them in as ſimple a manner as the nature of the Subject will admit of; if I have been conciſe, I doubt not I ſhall be excuſed, if I have but ſaid enough.

But, as I think I have ſaid enough, in this place, I ſhall ſtraightway proceed to the Subject; through which, if the Reader be inclined to follow, with an intention to learn what it contains, I am much

much miſtaken if he loſes his labour; and, for ſuch as peruſe it only with intent to cavil, I hope they will be greatly diſappointed, and find but little to cavil at. I am not ſo vain as to ſuppoſe it is without defect, or that it will pleaſe all, for that I know is impoſſible; but, if I have made it intelligible and uſeful to common capacities, it is what I aimed at, and ſhall reſt ſatisfied in the ſuppoſition that my labour is not entirely loſt.

The greateſt fault in Tacquet is, that his Figures are too ſmall and trivial; and it is a general fault, that they are often incorrect and frequently contradict the deſcription. Correct and well adapted Schemes are certainly of ſome conſequence, in which, I have been very particular; and have, alſo, carefully reviſed the letter preſs, ſo that, I hope there are but few errors have eſcaped my obſervation. For, Errors, in miſplacing the Referrences to the Schemes, and ſometimes omitting them entirely (which is better of the two) is unpardonable in mathematical Works; having frequently experienced, in moſt Authors that I have peruſed, the perplexity it occaſions; eſpecially, when the ſubject is quite new to the Student. But, if any ſhould remain unnoticed, I hope the candid reader will impute it to human fallibility, and not quarrel with it on that account, for I am certain he will not meet with many.

Although I have, in this Treatiſe, deviated greatly from Euclid, in many particulars, I have endeavoured to make it generally uſeful; by putting his Numbers after mine, to each Propoſition, and alſo by means of an Index, I have ſhewn where to find any Propoſition of Euclid; which, in caſe of reference, to Euclid, in other Works, may be readily turned to.

I have well conſidered and digeſted every Propoſition, have carefully reviſed them over and over with the ſtricteſt attention, and I am fully convinced, that there is no omiſſion of any thing that is eſſential, or neceſſary to be known. Notwithſtanding, I have abridged the whole Elements, particularly the firſt, the third, the fifth, and the eleventh Books; yet, I dare venture to affirm, that I have not omitted the ſubſtance of any Propoſition which will ever be referred to, by Authors in any other Science.

AN INTRODUCTION TO GEOMETRY,

CONtaining a full Definition of the Terms peculiar to, or made uſe of in that Science, with explanatory Notes and Remarks, where it is neceſſary to illuſtrate or enlarge.

Likewiſe a ſhort Theory of Plane Angles; in which they are more fully explained than in any other Work of the kind, that I have ſeen. Indeed, moſt Authors in Geometry are entirely deficient in that reſpect; for want of which, the young Geometrician is frequently at a loſs, to conceive a clear Idea of Angles. I have therefore, been explicit on that Head.

It alſo contains an Explanation of all the Abbreviations made uſe of in this Work; many of which are explained in Engliſh Grammars, and other ſchool Books, and ought to be known to every Engliſh Reader; yet, as I know that their true ſignifications are not ſo generally underſtood, it may not be thought impertinent, or ſuperfluous here.

Nor have I omitted any thing, that is neceſſary to elucidate the Subject I am about to treat on; at leaſt, I think I have not; for, in the courſe of my own ſtudy of it, and in teaching others, I have been enabled to diſcover what is needful to moſt Capacities; and if I have any pretence to merit in this Work, it is chiefly on that account; in rendering an intricate, yet generally uſeful and moſt neceſſary Science, attainable to ordinary Capacities; and, at the ſame time, I hope, not exceptionable to thoſe of acuter talents.

I flatter myſelf that it will not be leſs acceptable to any, for being eaſy to be attained; to write only for Proficients, is to little purpoſe. By ſuch I may, in ſome caſes, be thought

rather prolix; yet, I preſume, not tedious; a repetition is ſometimes neceſſary to young Minds, and is more agreeable, in general, than turning back, which they are too frequently obliged to do; it being impoſſible to retain, in memory, all that is paſſed over on the firſt peruſal.

GEOMETRY, according to its original derivation, ſignifies to meaſure the Earth. It is a Science which contemplates continued Quantity, Extenſion or Magnitude, abſtractedly conſidered; it teaches the nature and properties of Lines, Angles, Figures, Surfaces and Solids.

Geometry is in two parts, ſpeculative and practical; the firſt demonſtrates the properties of Right Lines, Figures, &c. in ſpeculation; from which is deduced the practical part for various uſes, for the benefit of mankind, in mechanic Arts, &c.

Euclid has judiciouſly divided the Subject into Books or Sections; each of which, treats of different Figures, or different properties of Figures, the power of Lines, Proportion, &c. which ſome Authors have thought proper to deviate from, though without any juſtifiable reaſon for ſo doing.

It treats, in the firſt, third, fourth, and ſixth Books, of Plane Figures, and thence is called Plane Geometry; and afterwards, in the 11th and 12th of Euclid, the 7th and 8th of theſe Elements, of Planes and Solids.

Each Book contains ſundry Propoſitions; from which, are deduced Corollaries, Scolia, &c. the ſignification of all which I ſhall firſt beg leave to explain or define; and then proceed to the Definitions of the more eſſential Terms, which are the Subject of Geometry.

A DEFINITION is the defining or explaining the full ſignification of any Term or particular Word, peculiar to, or made uſe of, in that Science of which we are about to treat.

A PROPO-

A PROPOSITION is either a Theorem, propoſed to be proved or demonſtrated, contemplatively; or, it propoſes ſomething to be done, problematically or mechanically.

A CONVERSE PROPOSITION is the contrary of the other; that, which in the foregoing was the Concluſion, drawn from the Premiſes of it,

e. g. If a Triangle have two equal Sides, the Angles which they ſubtend are alſo equal; the Converſe is, that if a Triangle have two equal Angles, the Sides ſubtending them are alſo equal.

A THEOREM is a ſpeculative Propoſition; a declaration of certain properties, equality, or other proportion relative to Quantity, or Figure, mathematically conſidered.

A PROBLEM, is a Propoſition which propoſes ſomething to be done, practically or mechanically.

An AXIOM is a ſelf evident Propoſition, which does not require to be demonſtrated. See Axioms, Book 1. El.

A LEMMA is a Propoſition, as it were by the bye, or out of the way, which ſerves, previouſly, to prepare the way for the more eaſily comprehending the Demonſtration of the following Propoſition.

I do not make uſe of Lemmas in this Work, as ſome geometrical Authors do; for if there be a neceſſity for a Lemma, I ſee no reaſon why that Lemma is not as much a Propoſition as any other. The 7th and 16th Propoſitions of the firſt Book of Euclid, may be called Lemmas, for they are certainly redundant Propoſitions. In other mathematical Works, Lemmas are frequently neceſſary, but, in Geometry, they are quite inconſiſtent.

A COROLLARY is a neceſſary conſequence deducible from ſome Propoſition, already demonſtrated.

A SCHOLIUM is a remark, or uſeful leſſon derived from the preceding Propoſition.

A POSTULATE is a petition or requeſt which is required to be granted. See Poſtulates, p. 21.

HYPOTHESIS. Whatever is ſuppoſed or premiſed, in a Propoſition, is called the Hypotheſis or Premiſes of it; from which ſome certain Conſequence is deduced, as affirmed, and afterwards demonſtrated, called the Theſis or Affirmation.

e. g. If a Right Line, cutting two Right Lines, makes equal Angles with them both, thoſe lines are parallel.

Here, the Hypotheſis is, if the Angles are equal; and, the Conſequence, that the Lines are parallel.

SUPPOSITION. In demonſtrating ſome Theorems, it is neceſſary to have recourſe, frequently, to ſuppoſe ſuch and ſuch things, which are not ſo in reality; by the abſurdity of the conſequences, ariſing from ſuch a ſuppoſition, a concluſion is drawn, and the Demonſtration is made evidently to appear.

Such kind of Demonſtration is called *reductio ad abſurdum*, i. e. proving it to be abſurd, or impoſſible to be on that ſuppoſition; which, not being direct and poſitive, is, to many, very unſatisfactory; yet, if rightly conſidered, is full, though not direct Demonſtration.

CONSTRUCTION, is the contriving or diſpoſing, geometrically, Lines and Figures, neceſſary for making the Demonſtration appear, clear and conſpicuous; and muſt always

always be made of ſuch Figures or Lines as are already well underſtood; the Properties, of which, being previouſly demonſtrated.

DEMONSTRATION. When any thing is propoſed, or affirmed in a Propoſition, the Caſe is firſt ſtated and prepared, by drawing ſuch lines, or forming ſuch a Conſtruction as is neceſſary; and it is afterwards demonſtrated; that is, the truth of the Aſſertion is made to appear, obvious, and without the leaſt doubt remaining; the performance or operation of which is called the Demonſtration.

The three laſt Terms being common words in the Engliſh Tongue, may, by ſome, be thought impertinent; but, notwithſtanding the common acceptation of them is almoſt univerſal, yet the application of them in Geometry requires to be explained.

DEFINITIONS.

Of the eſſential and operative TERMS.

The Terms of Art, to be defined in any Science, are Names, arbitrarily given, by the firſt Authors, or others, to certain Symbols, Figures, Marks or Characters, poſſeſſing certain properties or relations, in reſpect of figure, poſition, ſituation, &c.

The operative Terms are, generally, technical Words, peculiar to that Science, though perhaps applicable to others; which are not of common uſe in Language, or have a different ſignification.

The following Definitions are frequently referred to, hereafter, for illuſtratien or proof of what is advanced in the Propoſitions.

When any Figure, &c. which we are contemplating, is found to poſſeſs ſuch or ſuch properties, we affirm it to be ſuch a Figure, as anſwers to them; or, in contemplating any Figure, given in the Premiſes, we affirm that it has ſuch or ſuch properties, arbitrarily, by the Definition of it; and therefore, it requires no other Demonſtration.

DEF.

DEFINITIONS.

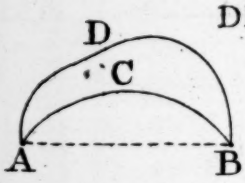
DEF. 1. A POINT is, rather to be conceived than underſtood to be, without dimenſions, therefore indiviſible.

In Plane Geometry, it is a Mark, ſuppoſed to be made with a ſharp-pointed Needle, or fine drawing Pen. As A.

A

DEF. 2. A LINE has length only, without other dimenſions, of breadth or thickneſs.

DEF. 3. A RIGHT LINE is the ſhorteſt that can be drawn between two given Points, (A and B) uſually called a ſtreight Line.



DEF. 4. A CURVE, or CURVED LINE, is any other than a Right Line; either regular, as ACB; or irregular, as ADB.

Curve Lines are of various kinds, as circular, elliptic, parabolic, &c. each of which has particular properties peculiar to them, and ſome in common.

DEF. 5. A SURFACE, or SUPERFICIES, is the outſide or external parts of Bodies, having no regard to thickneſs or ſubſtance; and therefore, has but two dimenſions, *viz.* length and breadth. As AB and CD.

Surfaces are various, as Convex, Concave, and Plane.

Irregular Surfaces are ſuch as are not, uniformly, any of the three, but may be compounded of them all.

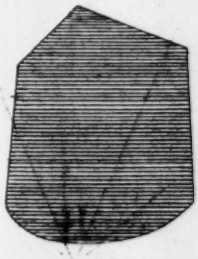
A convex Surface is one that is externally round or protuberant. Alſo, the outſide of the curve of a Circle, &c. is convex.

A concave Surface is one that is round internally, or hollow, ſuch as the inſide of a round Veſſel; the outſide is convex. The curve of a Circle towards the Center is concave.

DEF.

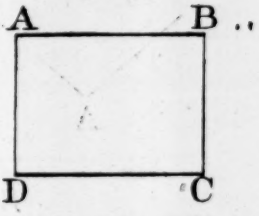
DEF. 6. A PLANE is a perfectly even and regular ſurface, which is neither convex nor concave, in any part; to which, if a Right Line or ſtreight Ruler be any how applied, it will touch in every Point.

Or, if any two Points, in a Plane, be joined by a Right Line, the whole line is in that Plane.



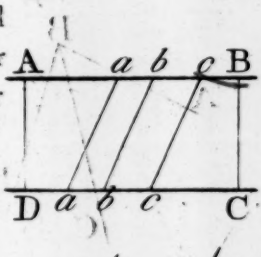
A Plane may be conceived to be generated by the direct motion of a Right Line, laterally; or whirled around on any Point in it.

If the Right Line AB be moved, directly to CD, there will be generated the Plane ABCD.



N. B. A Plane may be of any Shape or Figure, in reſpect of its bounds or limits.

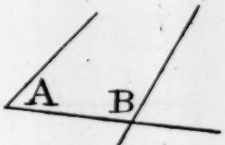
DEF. 7. PARALLEL. Right Lines are Parallel to each other, which, if produced infinitely, either way, and being in the ſame Plane, would never meet. As AB and CD.



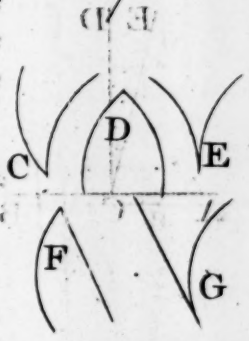
N. B. Parallel Lines are equidiſtant in every part; between which all Right Lines, as *a a*, *b b*, and *c c*, being alſo parallel, are equal.

But, the Diſtance between two parallel Lines is meaſured by a Perpendicular, AD or BC. See Perpendicular (Def. 10.)

The ſame holds true in parallel Planes.



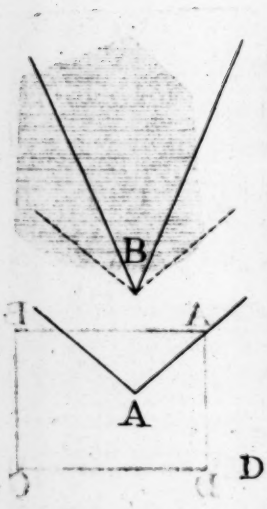
DEF. 8. A PLANE-ANGLE is formed by the meeting, touching, or cutting, of two Lines in the ſame Plane, ſo as not to fall into or conſtitute one Line.



Angles are either right-lined, as A or B; curved, as C, D and E; or mixed, as F and G; which are compounded of a Right Line and a curved or crooked one.

N.B. Angles are neither increaſed nor diminiſhed by the length of the Lines which form them; for, the Angle A is greater than the Angle B, notwithſtanding the Lines which form the Angle B are longer than thoſe of the Angle A; but, the Lines forming the Angle B are more inclined to each other.

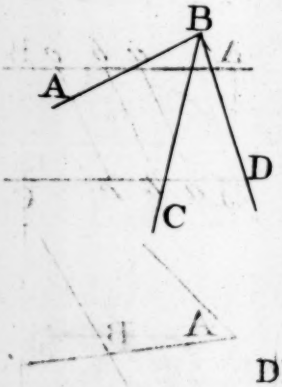
2. Ima-



2. Imagine the dotted Lines to repreſent the Angle A laid upon, or applied to the Angle B; it is evident, that the Angle B is leſs than the Angle A, having a greater inclination of its Sides, or the aperture between them leſs; therefore, a leſs Angle may always be contained in a greater.

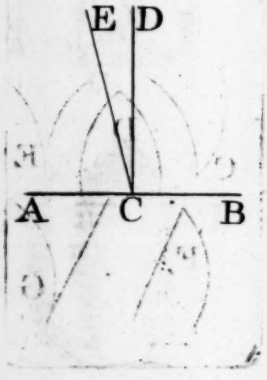
3. If the Sides of any Angle (as B) were lengthened infinitely, the Angle is not varied, if the inclination of the Lines remains the ſame; but, if you ſuppoſe them moveable, on the angular point, like a folding Rule, and are parted farther aſunder, approaching towards the dotted Lines, their inclination will then be leſs; and conſequently, the aperture or opening between them greater; and therefore, the Angle is ſaid to be greater.

DEF. 9. VERTEX, is the angular Point in which two Lines, forming a Plane Angle, meet and touch each other.



N. B. A ſingle Angle is uſually deſcribed by one Letter only, or other Character, as A or B; but, if three or more Lines meet in a Point, then three Letters are uſed, to ſpecify the Angle ſpoken of; as ABC, CBD or ABD; of which the middle Letter always denotes the Vertex; and means, the Angle made by the lines AB and BC, or CB and BD, &c. wherefore, the middle Letter, denoting the Angle, is underſtood to be twice named.

DEF. 10. PERPENDICULAR. A Right Line is perpendicular to another, when it does not incline to the other on either Side; but makes the Angle, on each Side, equal to the other.



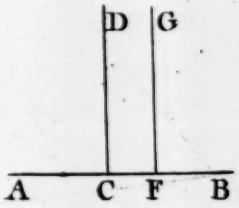
Thus; if the Angles ACD, DCB, are equal to one another, then CD is perpendicular to AB.

N. B. No more than one Perpendicular can be drawn from the ſame Point in a Right Line, on the ſame Side, and in the ſame Plane.

For, if any other Line, as CE, be drawn from the ſame Point, C, it cannot be a Perpendicular, but is ſaid to incline to AB; and, the inclination is on the ſame Side of the Perpendicular CD; i. e. the Angle ACE is the Angle of Inclination of the two Lines AB and CE.

N. B. 2.

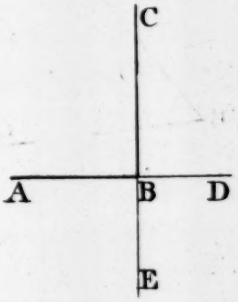
N. B. 2. If another Right Line, as FG, be drawn in the ſame Plane, from any Point F, perpendicular to AB, it will be parallel to CD.



DEF. 11. A RIGHT ANGLE is that which is formed by the meeting of two Right Lines which do not incline to each other, but either of them is perpendicular to the other.

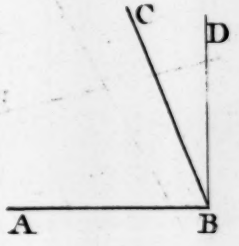
N. B. If either Side of a Right Angle be produced, or drawn out beyond the Vertex, there is neceſſarily generated another Right Angle. And, conſequently, if both Sides are produced, there will be generated four Right Angles.

Thus, ABC is a Right Angle; if, when AB or CB is produced, towards D or E, there is made another, CBD or ABE; and if both are produced, EBD is a fourth Right Angle.



DEF. 12. An ACUTE ANGLE is leſs than a Right Angle.

If the Line CB, meeting AB in the Point B, falls on this Side of a Perpendicular, BD, at that Point; the Angle ABC, being leſs than the Right Angle ABD, is therefore called Acute.

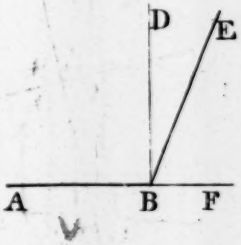


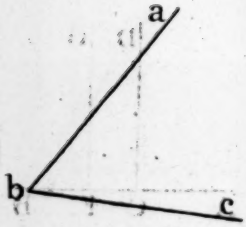
DEF. 13. An OBTUSE ANGLE is greater than a Right one.

If the Line BE falls on the other Side of the Perpendicular, BD; the Angle ABE is Obtuſe.

N. B. The difference, CBD, between an Acute Angle, ABC, and a Right one, ABD, is called the COMPLEMENT of the Angle ABC.

And, if either Side of an Obtuſe Angle, as AB, be produced, the Angle EBF is the Complement of the Obtuſe Angle; or its deficiency to two Right Angles, ABD, DBF.

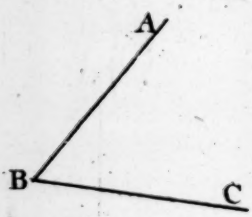




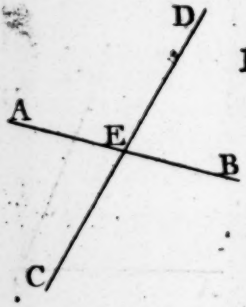
DEF. 14. EQUAL ANGLES. Angles are equal, when the Lines, which form them, have the ſame Inclination each to the other, reſpectively.

In the Angles a b c, ABC; if the Vertex, b, of the one, be applied to the Vertex, B, of the other, in ſuch wiſe, that the Side a b falling on A B, c b alſo falls on CB; then, there is the ſame inclination of c b to a b as of CB to AB, and the Angles a b c, ABC are equal.

N. B. The length of the Lines, or Sides, is not conſidered or regarded in the equality of the Angles, but only their inclination to each other.



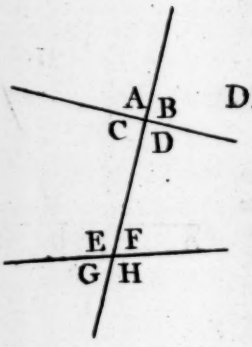
Angles have other Denominations, which are derived to them only from their Situation, in reſpect of each other; yet ſtill retain the general appellation of Right, Acute, or Obtuſe Angles. Such are the following.



DEF. 15. VERTICAL and CONTIGUOUS ANGLES.

If two Lines, AB and CD, cut and croſs each other, there are made four Angles, at the Point, E, of their mutual Interſection; either two of which, AED, CEB, or AEC and DEB, touching at their Vertices, only, are called Vertical Angles.

Any other two, as AEC, AED, or AEC and CEB, &c. having one Side, C E or A E, common to both Angles, are called Contiguous or Adjoining Angles.



DEF. 16. ALTERNATE ANGLES, and others.

If a Line croſſes or interſects two Lines, there are made eight Angles, A, B, C, D, &c.; of which C and F, alſo E and D, between the two Lines, one on each Side of the cutting Line, are called Alternate Angles.

C and E, alſo D and F, are called INTERNAL ANGLES on the ſame Side.

E and A, F and B, C and G, or D and H are called INTERNAL and OPPOSITE ANGLES, on the ſame Side.

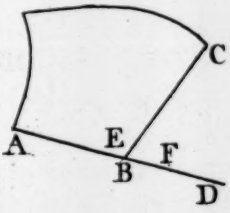
DEF.

DEF. 17. A PLANE FIGURE is a Space bounded on all ſides by one or more Lines in the ſame Plane.

2. Any Line, between two adjacent Angles, forming or bounding a Figure, is called a SIDE of that Figure.

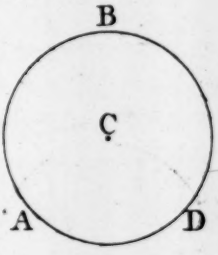
N. B. If a Plane Figure be bounded by Right Lines, only, it is called a Right-lined Figure; as Z. And if it be formed of Right Lines and curved, it is called a mixed Figure. As AC.

DEF. 18. INTERNAL and EXTERNAL ANGLES.



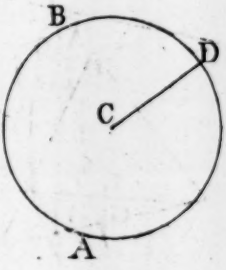
If any Side of a Plane Figure be drawn out beyond the Figure, as AB to D, the Angle E, or AEC, within the Figure, is Internal; and the Angle F or CBD, without the Figure, is called an External Angle.

DEF. 19. A CIRCLE is the ſimpleſt and moſt perfect of all Plane Figures, therefore the firſt; it is bounded by one regular and uniform curved Line, falling again into itſelf; which is called the CIRCUMFERENCE of the Circle.



The curved Line ABD is the Circumference; the Space, included within it, is the Circle.

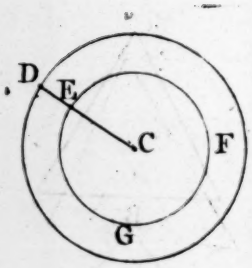
DEF. 20. CENTER of a Circle, or CENTRE, is a Point in the middle of a Circle, or the middle Point of a Circle; which is equally diſtant, every way, from the Circumference. As C.



N. B. The geneſis of a Circle is thus defined.

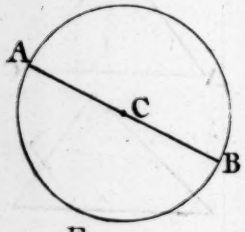
If a Line, CD, be conceived to be revolved quite around, on one extreme, C, fixed to a Pin or Point; the other extreme, D, will, in its revolution, deſcribe the Circumference of a Circle, ABD; and the Line CD, having gone over the whole ſpace, has generated the Circle, bounded by that Circumference.

Hence, it is evident, that all Right Lines, drawn from the Center of a Circle to the Circumference, are equal.



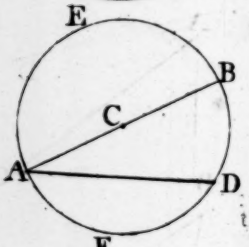
N. B. 2. Equal Right Lines deſcribe equal Circles; but, if another Line, CE, or, if any Point, E, be aſſumed in the Line CD, another Circle will be deſcribed, on the ſame Center, and in the ſame ſpace of time; whoſe Circumference, EFG, is to that of the other Circle, deſcribed by the Point D, in proportion to the Lines CE and CD, by which they were generated.

DEF. 21. DIAMETER, of a Circle, is a Right Line drawn through the Center, and terminated on both ſides by the Circumference. As A B.



2. Half the Diameter of a Circle is called the RADIUS. As AC or CB.

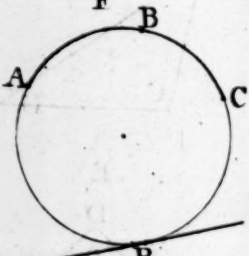
N. B. Every Diameter divides the Circle, and alſo the Circumference, in two equal Parts.



DEF. 22. A SEGMENT of a Circle, is any portion cut off by a Right Line; which is called a CHORD or SUBTENSE.

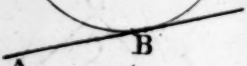
As AD, making two Segments, AED and AFD.

N. B. A Diameter is alſo a Chord Line.



DEF. 23. A SEMICIRCLE is a Segment, made by a Diameter, AB. As AEB or AFB.

Therefore, the Segment AED, which is greater than a Semicircle, is called a greater Segment; and AFD a leſſer Segment.



DEF. 24. An ARK, or ARCH, is any portion of the Circumference of a Circle. As AB, BC, or ABC.

DEF. 25. A TANGENT is a Right Line drawn without a Circle, and touching it in a Point only; which is called the POINT OF CONTACT.

As A B, touching the Circle in B.

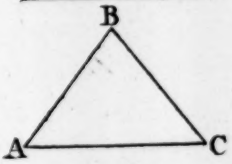
DEF. 26. A TRIANGLE is a Plane Figure bounded by three Right Lines, and contains as many Angles.

N. B. Not

N. B. Not leſs than three Right Lines can include a Space and form a Figure; wherefore, a Triangle is the firſt of all Right-lined Figures.

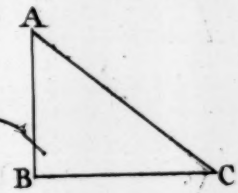
Triangles are of various kinds. As follows.

DEF. 27. 1. An EQUILATERAL TRIANGLE has all its three Sides equal, to one another.



DEF. 28. 2. An ISOSCELES TRIANGLE has only two equal Sides. AB and BC.

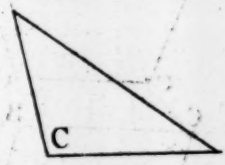
DEF. 29. 3. A SCALENE TRIANGLE has all its Sides unequal.



DEF. 30. 4. A RIGHT-ANGLED TRIANGLE is one that has a Right Angle. B.

2. The Side A C, oppoſite the Right Angle, is called the HYPOTHENUSE.

DEF. 31. 5. An OBTUSE-ANGLED TRIANGLE is one which has an Obtuſe Angle. C.

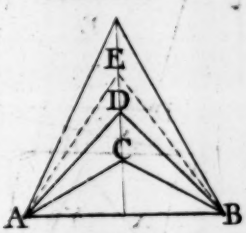


N. B. The two laſt are not diſtinct ſpecies of Triangles, but only a particular kind; which ſtill come under the general Denomination of Iſoſceles or Scalene.

An Iſoſceles or Scalene Triangle may be either right or obtuſe angled, or have all its Angles acute.

The Triangle ADB is right-angled; ACB is obtuſe-angled; and AEB has all its Angles acute; yet, they are all Iſoſceles.

So likewiſe, the Figures of the two laſt Definitions are Scalene Triangles.



DEF. 32. A QUADRILATERAL or QUADRANGLE is a Plane Figure which has four Sides, and four Angles.

Theſe are ſynonimous Terms; the firſt expreſſing it by the number of its Sides, the other by its Angles.

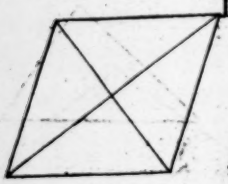
DEF. 33.

DEF. 33. A PARALLELOGRAM is a Quadrilateral, whoſe oppoſite Sides are parallel.

DEF. 34. A RECTANGLE is a Parallelogram, whoſe Angles are all Right ones. As X.

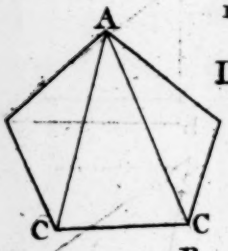
DEF. 35. A SQUARE is a Rectangle, whoſe Sides are all equal, to one another. Z.

N.B. All Rectangles and Squares are Parallelograms.

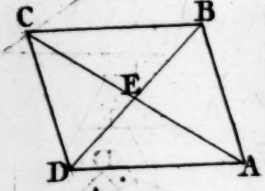


DEF. 36. A RHOMBUS is a Parallelogram, whoſe Sides are all equal, and its Angles not Right ones.

2. If the Sides of a Parallelogram are not all equal, and the Angles not Right ones, it is called a RHOMBOIDES.

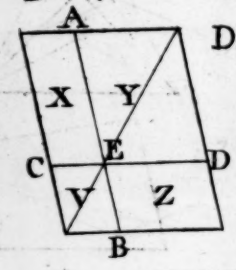


DEF. 37. A DIAGONAL is a Right Line drawn between any two Angles, that are oppoſite, in any right-lined or mixed Figure; i. e. from one Angle to the other. As AC.



N. B. In Parallelograms, the Diagonal is uſually called a Diameter; becauſe it paſſes through the Center (the middle Point, E, where the two Diagonals, AC and BD, interſect) and, as in a Circle, it divides the Parallelogram into two equal Parts.

Any Right Line, cutting a Parallelogram through its Center, is a Diameter.

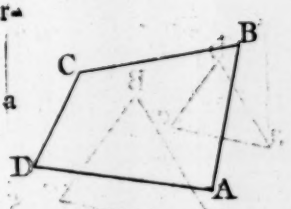


DEF. 38. COMPLEMENTS of a Parallelogram.

If any Point, as E, be taken in the Diagonal of a Parallelogram, and, through that Point, two Right Lines are drawn parallel to the Sides, both ways (AB and CD) it will be divided into four Parallelograms, V, X, Y, and Z; the two, X and Z, which touch the Diameter, in the Point E, only, are called the COMPLEMENTS; which, with either of the other, about the Diameter, taken together (as XYZ or XVZ) is called a GNOMON.

DEF. 39.

DEF. 39. A TRAPEZIUM is an irregular four-ſided Figure.



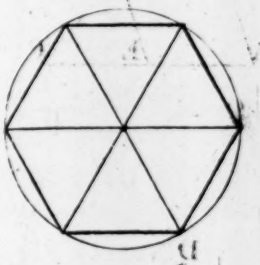
Wherefore, every Quadrilateral which is not a Parallelogram, is, conſequently, a Trapezium. As ABCD.

DEF. 40. A POLIGON. All right-lined or other Plane Figures, having more than four Sides, have the general appellation of Poligons, ſignifying many Sides.

N. B. Ordinate or regular Poligons are ſuch as have all their Sides and Angles equal; about which a Circle may be circumſcribed, whoſe Circumference ſhall paſs through every Angle of the Poligon; and, a Circle may alſo be inſcribed, which ſhall touch every Side.

Poligons have various Names, derived to them from the Number of their Sides; as follows.

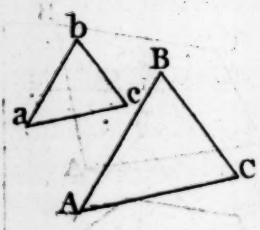
1. A PENTAGON is one that has five Sides.
2. A HEXAGON has ſix Sides.
3. A HEPTAGON has ſeven Sides.
4. An OCTAGON has eight Sides.
5. A NONAGON has nine Sides.
6. A DECAGON has ten Sides.
7. A DUODECAGON has twelve Sides.
8. A QUINDECAGON has fifteen Sides.



These eight are the moſt eſſential, in Geometry, and the moſt uſeful amongſt Mechanics. To ſpecify every kind of Poligon would be infinite.

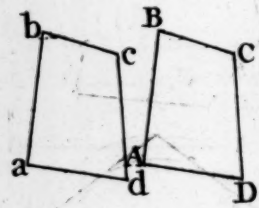
DEF. 41. PERIMETER is the ſum or meaſure of all the Sides of a Poligon, or other right-lined or mixed Figure, in one Sum; which is ſometimes called its Circumference, or PERIPHERY.

DEF. 42.



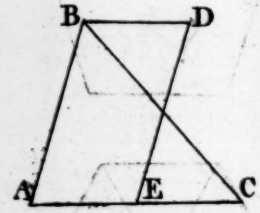
DEF. 42. EQUIANGULAR FIGURES are ſuch as have an equal number of Angles; and the Angles of one alſo equal, reſpectively, to the Angles of the other, each to its correſponding Angle.

As, a equal A, b equal B, and c equal C.



DEF. 43. CONGRUOUS FIGURES are ſuch as have all their Angles equal, reſpectively; alſo, the Sides, which contain equal Angles, or which are between equal Angles, are equal.

If the Angle a be equal A, b equal B, c equal C, and d equal D; and, if the Side ab is equal AB, bc equal BC, cd equal CD, and ad to AD; then, the Figures abcd, ABCD, are congruous.



DEF. 44. EQUAL FIGURES are ſuch as have an equal Area. See Def. 47.

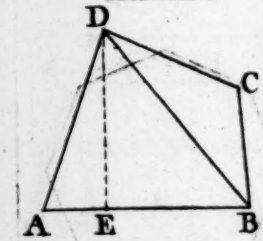
N. B. If two Figures of different Denominations have an equal Area, they are called equal Figures.

So, the Parallelogram ABDE is equal to the Triangle ABC.

Congruous Figures are equal and ſimilar.

DEF. 45. BASE, of a Plane Figure, is the Side on which it is ſuppoſed to ſtand erect. As AB.

N. B. It may be any Side, at diſcretion; but is generally applied to the lower Side, or that which is next towards us.



DEF. 46. ALTITUDE, of a Figure, is its perpendicular height from the Baſe.

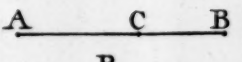
As ED, perpendicular to the Baſe AB, is the Altitude of the Trapezium ADCB; or of the Triangle ADB.

DEF. 47. AREA of a Plane Figure, or other Surface, is its ſuperficial Contents; i. e. the meaſure, or quantity of Space contained within the bounds of the Figure, expreſſed in ſquare Feet, Yards, or any other known meaſure, of length.

DEF. 48.

DEF. 48. SEGMENT of a Line, is any portion of a Line.

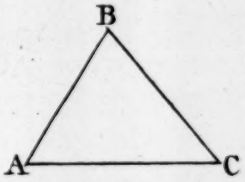
As AC, or CB, of the Line AB.

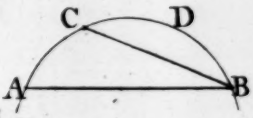


DEF. 49. SUBTEND. The Side of a Triangle which is oppoſite to any Angle is ſaid to ſubtend that Angle.

Thus, AB ſubtends the Angle C, AC ſubtends the Angle B, and BC ſubtends the Angle A.

So likewiſe, the Chord or Subtenſe AB, ſubtends the Ark ACDB; and CB ſubtends the Ark CDB.

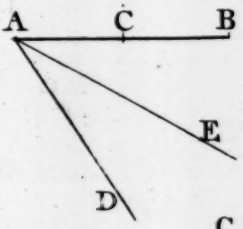




DEF. 50. To BISECT, is to cut or divide a Line or Angle, &c. into two equal Parts.

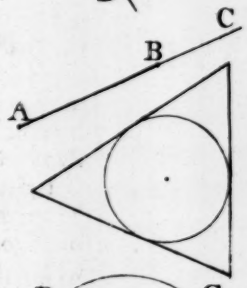
Thus AB is biſected in the Point C; and the Angle BAD, is biſected by the Line AE.

2. To TRISECT, is to cut, equally, into three Parts.



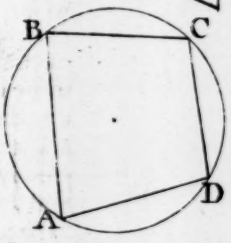
DEF. 51. To PRODUCE, is to draw out a Line or Plane, or to lengthen it at pleaſure.

Thus, the Line AB is produced to C.



DEF. 52. To DESCRIBE, is to draw a Line, Circle, or other Figure.

DEF. 53. To INSCRIBE, is to draw a Figure touching every Side of another Figure, internally; or, whoſe Angles ſhall all touch the Circumference of a Circle.



DEF. 54. To CIRCUMSCRIBE, is to deſcribe a Circle or other Figure paſſing through all the Angles of another Figure. As ABCD.

DEF. 55. EXTREMES or BOUNDS, are the utmoſt Limits of any thing.

So, the extremes of a Line are Points; the bounds or extremes of a Surface are Lines, (except the ſurface of a Sphere, which has no Bounds) and, the extremes of a Solid are Surfaces.

A Sphere has but one Surface; alſo, a Circle is bounded but by one Line, without beginning or end; conſequently it has no Extremes.

Some Readers will, perhaps, think I have already deviated from the Plan propoſed in the Preface, which was, to abridge the Elements of Euclid; but I hope they will find, that the time ſpent on the Definitions (by way of Introduction) is not thrown away; having made ſeveral of them uſeful Leſſons, as well as defined the Terms; and I am perſuaded, that the more perfectly they are underſtood, the progreſs, in the Subject they relate to, will be greatly facilitated. But, I muſt inform the Reader, that it is not the length of the Definition itſelf, but the number of Terms I have defined, and the Notes to ſeveral, which have ſwelled the bulk of them; having near twice the Number which ſome Authors have, and, in my opinion, not one ſuperfluous.

However, not to diſcourage the young Student, I adviſe him not to burthen his memory too much at once; for there is not the leaſt neceſſity that he ſhould retain them all, before he proceeds further; 'tis enough, at firſt, to read them over with attention, and underſtand them clearly as he reads; they will ſoon become familiar to him, by frequent uſe, in the courſe of his ſtudy of the Science.

I alſo particularly adviſe him, when he meets with any Term hereafter, of which he has not a clear Idea, to turn immediately back to the Definition of it; he may depend on it, that the time will not be entirely loſt.

Several of the Terms which I have here defined, being frequently uſed in the Subject of Geometry, are abſolutely neceſſary to be defined, but are not, properly ſpeaking, elementary. Such are the 4th, the 9th, the 14th, 15th, 16th, and 18th; the 36th, the 41ſt, 43d, 44th, 47th, and 48th, and the laſt ſeven; which are chiefly operative or practical Terms. Though I muſt freely own, that I ſee no reaſon why the 49th, 50th, 51ſt, and 52d, are not as neceſſary to be defined as the 53d and 54th: the 55th is made three ſeparate Definitions, in the 3d, 6th, and 13th of Euclid.

It is, or ought to be, the deſign of every Author on any Science, to make his Book a perfect Tutor; conſequently, no particular Term, made uſe of in that Science, and which is peculiar

to

to it, ſhould be left undefined: for, we ſhould ſuppoſe the Student to be entirely ignorant of all that relates to it. Can any thing be more abſurd, than to propoſe biſecting a Line or Angle, or producing a Line, &c. to talk of Alternate Angles, &c. of Diagonals, Complements, Subtenſes, &c. without having firſt defined what is meant by them? I have often been ſurprized at the omiſſion of the Definition of a Poligon, and the various kinds of Poligons; the forming, inſcribing, and circumſcribing of which, is the principal ſubject of the 4th Book. They are frequently called by the Terms Pentagon, Hexagon, &c. which have never been properly, if at all, defined. In the 23d Definition they are, in general, called many-ſided Figures; 'tis a ſtrange ungeometrical Term, and never once made uſe of after, but other Terms are aſſumed.

Euclid, himſelf, has not defined a Parallelogram, that moſt uſeful and neceſſary Figure. I admire Mr. Stone's Apology for that omiſſion, as being unwilling to increaſe the Number, unneceſſarily; and ſays, that Euclid has ſufficiently defined it in the 34th Propoſition. Mr. Stone indeed has, but not Euclid; for why elſe has Keill defined it before the Propoſition? But, if it be thought moſt eligible to define that Term in the Propoſition, where it is firſt made uſe of, why are not others ſo defined? why are not all, as well as this?

At the ſame time, Euclid has, in Mr. Stone's Opinion, given ſeveral unneceſſary Definitions; viz. the 3d, 6th, and 13th of his Euclid, which I always thought ſuperfluous; the 9th I think ſo too, alſo the 20th. He, likewiſe, thinks the 18th, 26th, the 32d, and 33d, uſeleſs. In reſpect of the 26th (the 29th of this) I cannot ſay it is eſſentially neceſſary; yet, as Scalene Triangle is a general Term, including all that are neither Equilateral nor Iſoſceles, I cannot think it redundant.

The 29th, of which he ſays nothing, is really ſuperfluous, viz. "when all the Angles are acute, it is called an acute-angled Triangle." I know of no properties peculiar to ſuch an acute-angled Triangle, that is not common to every Triangle; for every Triangle has, neceſſarily, two acute Angles. An equilateral Triangle is included in that Definition, and ſo are the Iſoſceles and Scalene, frequently.

Why is not a three-ſided Figure (Def. 21.) called a Triangle, as it is always called afterwards? we might as well go on, from four to five or fifteen ſided Figures. An Oblong (Def. 31.) for a Rectangle, is quite ungeometrical, and is never called ſo after. The Rhombus and Rhomboides, 32. and 33. (Def. 36. of this) are, in a great meaſure, uſeleſs, being fully ſignified in a Parallelogram, which alſo includes Squares and other Rectangles; all which, are only particular ſpecies of Parallelograms; the Rhomboides including all unequal ſided and acute angled.

I can by no means agree with Mr. Stone, in thinking the 18th Definition, of a Semicircle, needleſs; I alſo think, the Radius as neceſſary to be defined as the Diameter. In ſpeaking of Angles in a Segment of a Circle, Prop. 31. of the 3d Book, the 12. of this, viz. "the Angle in a Semicircle is a Right one," &c.; ſince there is a neceſſity for calling it ſomething, I cannot but think a Semicircle more elegant and expreſſive than half a Circle, which does not confine it to any particular Shape or Figure; provided it has half the Area, it is half a Circle; whereas, the Semicircle (which undoubtedly means half a Circle) is always underſtood to be contained under a Diameter and half the Circumference.

Mr. Stone is ſomewhat too dogmatical in his remarks. I cannot be of Opinion, that the manner in which Euclid has compiled his Elements, is the beſt that can poſſibly be, becauſe Euclid lived two thouſand Years ago. 'Tis not to be imagined that the mathematical Sciences had arrived at their *ne plus ultra* in his Days, for it is notorious they were not; why then, ſhould we ſuppoſe, the Elements of Geometry to be in their greateſt perfection? I am not ſo much wedded to antiquity, as to think them infallible; neither do I think that all Euclid's Definitions are neceſſary, or that he has omitted none that are ſo: Such, therefore, as are uſeleſs I have rejected, and have ſupplied their places with others, which I think more eſſential, and abſolutely neceſſary to be defined.

POSTU-

POSTULATES.

Poſtulates are fundamental Principles in any Science; which, being plain and ſimple in themſelves, may readily be granted; although, it is not poſſible to give perfect and indiſputable Demonſtration.

The following are, therefore, requeſted to be granted.

1. That there may be a perfect Plane; and that it may be extended at pleaſure.
2. That a Right Line may be drawn between any two given Points; i. e. from one Point to the other.
3. That a Right Line may be produced at pleaſure; i. e. that a finite Line may be continued or lengthened.
4. That a Circle may be deſcribed on any Center, and with any given Radius.
5. That one Figure may be applied to, or laid upon another Figure.

Theſe Poſtulates muſt be granted, at leaſt in Idea, or all Geometry falls at once to the Ground; for, if there cannot be a Plane, a Right Line, or a Circle, the whole Elements of Geometry are to no purpoſe; as it will be impoſſible to form a Conſtruction, whereby we may demonſtrate the moſt eſſential Properties of Figures in general, whether Plane Figures or Solids; and conſequently, if no Demonſtration can be given, there is an end of the Science, having no Data to build on.

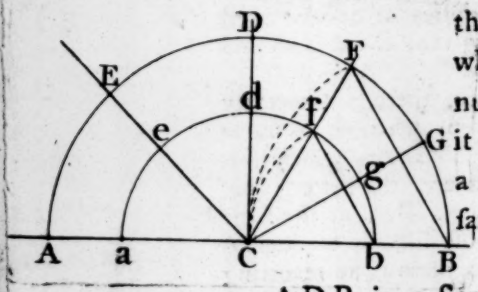
In every mathematical or phyſical Science, there is a neceſſity for ſome Data or firſt Principles to be given, whereon to frame Hypotheſes, in order to demonſtrate the Theorems which follow; and the more ſimple thoſe Principles are, the better; becauſe there will be leſs room to diſpute them. But, at the ſame time, if they are diſputed, they are the moſt difficult to demonſtrate, for being the moſt ſimple; becauſe, there is no reverting back to any thing more ſo; and conſequently, there can be no Demonſtration given. That the thing is ſo, of itſelf, is ſomewhat arbitrary, notwithſtanding there is no poſſibility of denying it; therefore, the more ſimple the firſt Principles are, the readier the aſſent will be given; and the Demonſtrations, of the moſt complex Propoſitions, which follow after, will be eaſier obtained and more firmly ſupported; and conſequently, the whole Science, which is built on thoſe Principles, is more ſolid and permanent, and more ſecurely eſtabliſhed.

THE THEORY OF PLANE ANGLES.

BEfore I proceed to practical Geometry, I think it proper, firſt to explain the Theory of Plane Angles; which I look on as a very neceſſary Introduction.

In order to have a perfect and clear Idea of Plane Angles, and to determine their Quantity by a certain ſtandard or meaſure, the Circumference of a Circle is ſuppoſed to be divided into 360 equal parts, called Degrees (anſwering to the number of the Degrees on the Equator) it is evident that thoſe diviſions will be leſs in a ſmall Circle than in a larger.

1. If from two Points in the Circumference of a Circle, as E and F, Lines are drawn to the Center, as EC and FC, there is made an Angle at the Center, C, which is greater or leſs, according to the number of Degrees on the Ark EDF, but it will be the ſame in a ſmall Circle as in a large one, i. e. the lines will have the ſame inclination to each other. e. g.



2. ADB is a Semicircle, whoſe Center is C; the Arch AEDFB contains 180 deg. half 360, the whole Circumference.

From the middle point, D, of the Arch ADB, which is 90 deg. each way, from A and B, if CD be drawn, it will be perpendicular to AB; for ACD and DCB, are Right

Right Angles, at the Center (Def. 11.) the Arks AD, DB, being each a fourth part of the whole Circumference, or half the Semi-circumference: hence, a Right Angle is ſaid to be of 90 Degrees.

3. If the Ark AD be biſected in E, and EC be drawn, the Angles ACE, ECD, will be each of 45 deg. half ACD a Right Angle, or 90 Degrees.

And, if the Ark DB be triſected, at F & G, (i. e. divided into three equal parts), and FC be drawn, the Ark DF containing 30, and FGB 60 deg. the Angle DCF is ſaid to be an Angle of 30, and FCB an Angle of 60 deg.

By which means, an Angle of any quantity may be obtained, or meaſured.

4. On the ſame Center, C, with any other Radius, as C a, let the Arch a e d f b be drawn, which is alſo a Semicircle.

It is very obvious, that it is alſo divided into the ſame number of parts, and in the ſame proportion, as the Arch AEDFB; for it is biſected in d, and a d is again biſected in e, and d b is alſo triſected at f and g; wherefore, AD, a d are each a fourth; ED, e d an eighth; BF, bf, a ſixth; and FD, f d, a twelfth part of their reſpective Circles; and the Angles ACD, aCd; ECD, eCd, &c. are the ſame in both.

From all which, it is clear, that, Angles may be formed or meaſured by an Ark or Circle of any Radius. And alſo, that equal Arks of the ſame, or of equal Circles, or an equal number of degrees in a Circle of any Radius, will form equal Angles at the Center.

5. If you would have an Angle of 60 degrees at the point C, of the line BC.

With

With any Radius, at pleaſure, deſcribe the Ark BD, cutting the Line BC in B; with the ſame Radius, on the Center B, cut the Ark BD at F, and draw CF.

It is very clear, that the Angle BCF would be the ſame, if a leſs Radius had been taken, as C b.

For, draw the Chord Lines FB and fb, each will be equal to the Radius of its reſpective Circle; and, the Triangles CFB, Cfb are equilateral; whoſe Angles are of 60 degrees each (Cor. 1. 9. 1.); for the Arks BF and bf are, each a ſixth part of the whole Circumference of their reſpective Circles, of which C is the common Center, (ſee Prop. 11. 4.); and conſequently, each contains 60 degrees on the circumference of that Circle, of which it is a Part; whoſe Radius is CB or Cb.

A deſcription of the Inſtrument called a PROTRACTOR, with the application of it, in meaſuring and making Angles, of any known quantity or meaſure, may not be improper in this place. It is of ſpecial uſe in Surveying, in drawing Plans of any piece of Ground for building on, &c. or of Buildings, already erected, being readier and more exact than a Line of Chords.

The Protractor is a Semicircle, divided on its Limb or Semi-circumference, AEDFB, into 180 equal parts; having a ſmall Notch at C, the Center. See the laſt Figure.

Some Protractors have a Scale, added to the Semicircle, which are the beſt and readieſt in uſe.

In meaſuring an Angle, apply the Diameter, i. e. the Edge or Right Line AB, to either Side of the Angle ACE, with the Vertex, C, of the Angle, at the center of the Protractor; and, where-ever the Side CE, cuts the Limb or circular edge of the Inſtrument, obſerve how many Degrees there are from A to E, the Ark intercepted between the Sides AC and CE, of the Angle ACE.

If

If it contains 45 or 50, or whatever number of Degrees it happens to be, (as 45 by the Figure) the Angle ACE is of ſo many Degrees. If it had cut the Arch at D, as ACD, it is a Right Angle; and if beyond D, as ACF, it is obtuſe; the Complement FCB, i. e. the Ark FB, being ſubtracted from the Arch of the Semicircle, ADB, or 180 degrees, gives the quantity of the Angle ACF.

2. If it is required to lie down or make an Angle of ſome known Quantity (as 45 deg.) at the Point C, of the Line AB.

Apply the edge of the Protractor, as above, with the Center, C, at the Point given; make a Mark or Point at E; take away the Inſtrument, and draw E C.

Thus, may any Angle whatever be laid down on Paper.

A Scale of equal Parts is nothing more than a Right Line divided into any number of equal Parts, at pleaſure.

Each Part may repreſent any meaſure you pleaſe, as an Inch, a Foot, a Yard, &c.; for, being equal, whatever meaſure any Object or Figure contains, in length and breadth, a ſimilar Figure may be conſtructed, on a Plane, having or containing the ſame number of Diviſions, each way, on the Scale, as the real Object contains of Feet, Yards, &c. One of thoſe Parts is generally ſubdivided into parts of the next inferior denomination, or into tenths and hundredths, denoting the Decimal Parts.

Obſerve, that the Diviſions on the Scale, (whatever meaſure is repreſented by them,) muſt always be adapted to the Proportion you would delineate any Object, or form a Deſign.

See the Appendix (Page 15 and 16) for the conſtruction of Scales.

N. B. A pair of Compaſſes or Dividers, a Drawing Pen, and a ſtreight Ruler, are all the Utenſils that are requiſite, in Plane Geometry.

The Board or Paper, on which we draw any geometrical Figure, is ſuppoſed to be a Plane.

ABBREVIATIONS &c. EXPLAINED.

&c. *Et cætera.* And all the reſt. When, what is ſuppoſed to follow may be readily underſtood.

i. e. *Id eſt.* That is. When, what has been ſaid requires to be further explained.

viz. *Videlicet.* To wit; or, that is to ſay. When any thing advanced is given in Groſs, which is more particularly ſpecified, as follows after.

e. g. *Exempli gratia.* For inſtance; or, for the ſake of example. When an Example is to be given of what is advanced.

N. B. *Nota Bene.* Mark well. That is, take particular notice of that Paragraph.

Q. E. F. *Quod erat faciendum.* Which was required to be done.

Q. E. D. *Quod erat demonſtrandum.* Which was to be demonſtrated.

DEM.	Demonſtration.	Hyp. AB.	Hypothenuſe AB.
COR.	Corollary.	Perp.	Perpendicular.
SCHOL.	Scholium, or remark.	Dia.	Diameter.
APPL.	Application, or uſe.	Diag.	Diagonal.
Par. AB. or P. AB.	Parallelogram AB.	Trap.	Trapezium.
		Pent.	Pentagon, &c.

Rect. AB. Rectangle AB, or ABCD, &c.

R. Ang. ABC. Right Angle ABC, &c.

Line AB. Denotes Right Line AB, &c.

Note. When it is required to join any two Points, it is meant, that a Right Line be drawn between them, i. e. from one Point to the other.

AB-

ABBREVIATIONS, by way of Reference.

Poſt. 1. or 2. Refers to the firſt or ſecond Poſtulate; where it is requeſted that ſuch or ſuch things may be granted.

Def. 6. or 7. Refers to the ſixth or ſeventh Definition in the general Introduction. But, when there are two numbers, it refers to the 6th Definitions of the 3d or 5th Book.

Or, Def. 6. 3. or 6. 5. &c.

Ax. 3. Refers to the third Axiom for Demonſtration, ariſing from ſelf-evident properties of things.

2. Th. P. A. Refers to the ſecond Article in the Theory of Plane Angles, for illuſtration or Proof.

Pr. 1. or 2. &c. Refers to the firſt or ſecond Problem, for the conſtructing of ſome Figure, &c.

P. 1. or 2. &c. Refers to the firſt or ſecond Propoſition of that Book, for proof of the Aſſertion.

P. 2. 3. &c. To the ſecond Propoſition of the third Book.

C. 2. 4. 1. To the 2d Corollary, of the 4th Propoſition of the firſt Book.

Hyp. That the thing is ſo by the Hypotheſis, or is given in the Premiſes.

Sup. That it is ſo by Suppoſition, only.

Con. That it is ſo by Conſtruction; i. e. the thing was formed or made ſo.

Conſ. conſequently. Th. therefore. Wh. wherefore, it is ſo.

Note. When there is but one Number within Parentheſis or otherwiſe, as (15.) &c. it refers to the fifteenth Problem or Theorem of that ſame Book. But if there be two Numbers, as (10. 1.) or (12. 3.) &c. it refers to the 10th Prop. of the firſt Book, or to the 12th of the third, &c.

INSTRUCTIONS FOR YOUNG STUDENTS.

I would adviſe the young Practitioner to draw every Figure as he proceeds; carefully remarking what things, as Points, Lines, Angles, &c. are given; which are, in general, ſtronger marked than the operative Lines; they being either dotted or finer drawn; the given Lines, &c. are, by that means, obvious and diſtinguiſhable from the other.

In Geometry, as in Arithmetic, there is always ſome Data or things given; from which, in Theory, other Properties are deduced, as a neceſſary conſequence; and, in Practice, ſomewhat is required to be done, or performed, from what is given.

Let the Practitioner, therefore, ſelect the given things, and mark them down, firſt, in the poſition given in the Premiſes; but, with as much variation as it will admit of; i, e. he need not put them exactly as in the figure, only obſerve that they are as required.

e. g. In Prob. 4. an Angle is required to be made at the extremity of a given Line; but the poſition of that given Line is not abſolutely determined; alſo, the Angle may be made at either extreme, and either above or below the given Line.

Likewiſe, in the 6th and 7th Problems, the Perpendicular may be drawn, and the Point, in the 7th, given on either Side of the Line; for, let it be obſerved, and carefully remembered, that, by the Term Perpendicular, nothing more is meant than the Poſition one Line has to another; which Poſition, is when they make a Right Angle or Right Angles with each other; no regard being had to the poſition or ſituation of either, ſeparately.

Theſe things being premiſed, and the given Lines, &c. deſcribed on Paper; carefully obſerve the directions given in the operation, and proceed accordingly, ſtep by ſtep, drawing every Line, Angle, Ark, &c. as the Problem directs.

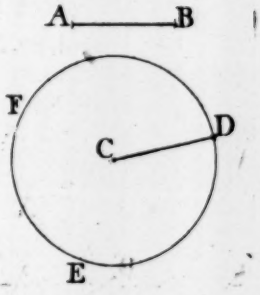
PRAC-

PRACTICAL GEOMETRY.

PROBLEM I.

To deſcribe a Circle of any given Radius, and on a given Center.

AB is the Radius given, and C the given Center.



FIX one Point of a pair of Compaſſes in either extreme of the given Line, AB, and extend the other Point to the other extreme, i. e. open the Compaſſes equal to the given line.

Then, fix one Point of the Compaſſes in C, the Center given, and revolve the other Point around; which, by its revolution, will deſcribe the Circumference DEF. - - - - - Poſt. 4.

DEF contains the Circle required (Def. 19.)

For, if a Right Line be drawn from the Center to the Circumference, as CD, it is equal to the given Line AB; by Conſtruction.

PRO-

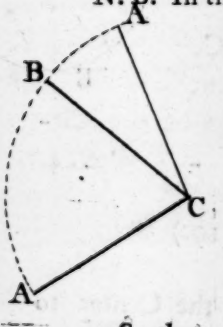
PROBLEM II. 2nd. I. Euclid.

To draw Right Line from a Point given, C, which ſhall be equal to a given Line; AB.

Extend or open the Compaſſes equal to the Line given; fix one of their Points in C, the given Point, and, with the other, deſcribe a Circle, or portion of a Circle only, at D. (Pr. 1.)

Apply a ſtreight Ruler cloſe to the Point C, and croſſing the Ark at D, (in whatever poſition you require the Line to be) and, with the point of a Pencil or a Drawing Pen (applied firſt to the Point C, and drawn, along the edge of the Ruler, to the Ark at D) deſcribe the Right Line CD. (Poſt. 2.) Which is equal to the given Line AB; by Conſtruction; and by N. B. Def. 20.

SCHOL. Thus, the geneſis of a Right Line (Def. 3.) is conceived to be by the direct motion of a Point.



N. B. In the practice of Geometry, it is often required to draw or to make a Line equal to another Line given; which is done by drawing an Ark of a Circle, as AB, from the given Line, AC, till it cuts the other; if the two Lines, AC and CB, touch at the Point given, C, which is made the Center.

But, if they do not touch, the Line given is taken for Radius (as in the Problem) and an Ark drawn where it is required; for equal Circles have equal Radii, as well as all Radii of the ſame Circle are equal; which needs no other Demonſtration than the geneſis of a Circle, in N. B. Def. 20.

So that, hereafter, when two Lines are obſerved to be Radii of the ſame Circle, it is ſufficient Demonſtration that the Lines are equal; and alſo, when they are made Radii of equal Circles.

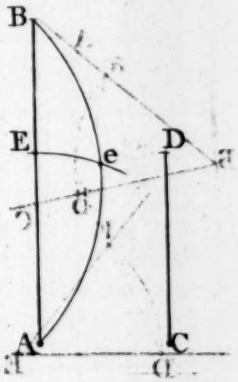
APPL. The Application of this Problem, in deſigning, is to delineate or draw, on Paper, &c. a Right Line equal to ſome known meaſure, as AB, by a Scale of equal Parts.

PRO-

PROBLEM III. 3. I. Euclid.

A Right Line being given, to cut off a portion or ſegment equal to another given Line, or known meaſure.

AB is the firſt given Line, and CD the meaſure of the Segment required to be cut off.



Extend the Compaſſes from C to D, i. e. with the Radius CD, ſetting one Point of the Compaſſes in either extreme of AB, (from which the given Segment is required to be cut off), as A, with the other Point, draw a ſmall Ark cutting the Line AB, at E. Q. E. F.

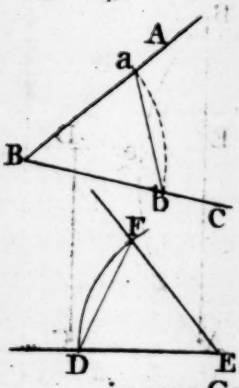
2. After the ſame manner, any portion of an Ark of a Circle may be cut off. As AeB.

APPL. In delineating, the given Line AB may be ſuppoſed to be an indefinite Line already drawn; or, it may repreſent a certain meaſure, by ſome Scale, ſuppoſe 5 Feet, being made equal to 5 Diviſions on the Scale; and it is required to cut off 3 Feet from the extreme Point A, of the Line AB, or two Feet from the other extreme, B.

By which means, a Right Line may be divided in any Proportion required.

PROBLEM IV. 23. I. Euclid.

To make an Angle, equal to a right-lined Plane Angle given.



ABC is the given Angle, and DE a Line given; it is required to make, at the point E, an Angle, with the Line DE, equal to the given Angle.

With any Radius, at diſcretion, on the Vertex of the given Angle, B, deſcribe an Ark, a b, cutting the two Sides AB and BC in the Points a and b. - - - - Poſt. 4.

With the ſame Radius, on the Point E, draw the Ark DF; take the meaſure of the Ark ab in your Compaſſes; make DF equal ab. - - 2. Pr. 3. Draw EF through the Point F; and it is done; Q. E. F.

The Angle DEF is equal to the given Angle ABC; i. e. FE inclines the ſame to ED, as AB to BC.

This is evident, from the Theory of Plane Angles, Art. 4, a b and DF being equal portions of equal Circles.

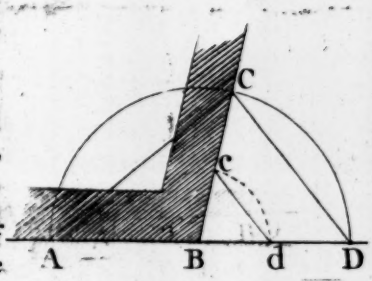
Or, by drawing the Chord Lines ab and DF, the Triangles Bab, EDF, are Congruous. - - - - Con. And the Angle DEF is equal ABC. - - P. 7. 1.

APPL. From this Problem we learn to delineate, i. e. to lay down or draw, on Paper, any right-lined Angle, of a piece of Ground or Building, &c. which we have meaſured, to form a Plan of it,

The

The Angle may be taken in the manner following.

If it be an internal Angle, as CBD; ſet off equal meaſure, from B, to C and D, and draw, or meaſure only, the Diagonal, or Chord CD.



Suppoſe you have made BC and BD each equal 5 feet, and the Chord CD meaſures 6 feet 6 inches, or as it happens; then, by a Scale of equal Parts, having drawn BD, at diſcretion, take 5 diviſions off the Scale; which may be either inches, half inches, or any other meaſure, provided they are equal.

With that Radius, on B, draw the Ark CD, cuting BD in D.

Take $6\frac{1}{2}$ of the ſame diviſions in your Compaſſes; and, ſeting one Point in D, make a ſmall Ark, at C, with the other, and draw BC.

So ſhall CBD be an Angle laid down, equal to the Angle which was meaſured.

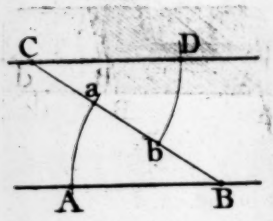
For (by N.B. Def. 8.) the length of the Lines or Sides makes no difference in the Angle; wherefore, if the Lines BC and BD were produced, equal to the Originals; i. e. equal 5 feet each; then would CD meaſure 6 feet 6 inches, and the Angle CBD would remain the ſame; which is obvious, if Bc and Bd be taken a half or a third part, or any other portion of BC or BD; for then, cd will be the ſame portion of CD; viz. a half or third, &c.

But, if ABC be the external Angle of a Building, ſo that, by reaſon of the obſtruction of the Walls, &c. we cannot obtain the meaſure of the Chord Line, AC, it muſt then be got by its Complement of two Right Angles; i. e. by producing one Side, as AB, to D, and proceeding as before.

For, having aſſumed the Point B in the Right Line AD, and made the Angle CBD equal to the Complement of two Right Angles; the remaining Angle, ABC, is the inacceſſible Angle required.

PROBLEM V. 31. I. Euclid.

To draw a Line through a given Point, which ſhall be parallel to a Right Line given.



AB is the given Line, and C is the Point given.

Through C, draw at pleaſure a Right Line, as CB, cuting the given Line, AB, in the point B.

With any Radius, on B, deſcribe the Ark Aa; and, on C, the Ark bD, with the ſame Radius.

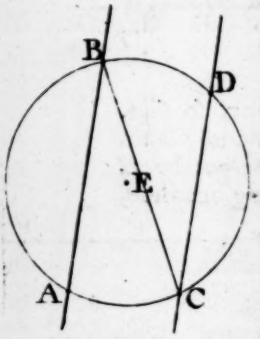
Make bD equal to Aa. - - - - Pr. 3.

and through the Points C and D, draw a Right Line.

Which will be parallel to the given Line AB. Q. E. F.

For, the Alternate Angles ABC and BCD are equal. Con.

Therefore, CD is parallel to AB. - - - P. 4. 1.



Otherwiſe, thus.

With any Radius, at diſcretion, ſet one Point of the Compaſſes in the given Point, C, and fix the other, at pleaſure, in E, either in the Line or out of it; on which Center deſcribe a Circle, paſſing through C; or draw the Arks AC and BD only.

Make BD equal AC, and draw CD;

Then is CD parallel to the given Line AB. Q. E. F.

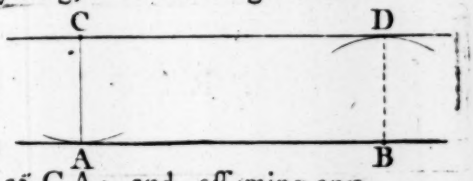
For, let CB be drawn.

Dem. Then, the Angles ABC, BCD are equal. Cor. 10. 3.

Therefore AB is parallel to CD. - - - P. 4. 1

Take

Or, when the Lines are very long, the following method is the moſt eligible.



Take the diſtance of the given Line AB, from the given Point C, in the Compaſſes; as CA; and, aſſuming any Point, B, towards the other end of the Line, deſcribe an Ark, at D, with that Radius.

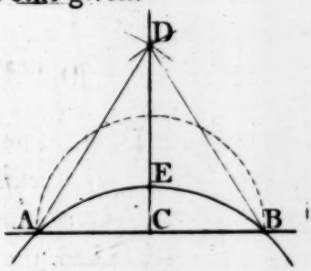
Apply a Ruler to the Point C and the Ark at D, and draw CD; which will be parallel to AB. - - Def. 7.

For, the Perpendiculars, AC and BD, being drawn, are equal, by Conſtruction.

PROBLEM VI. 11. L. Euclid.

To draw a Perpendicular, from any Point, in a given Right Line.

AB is the given Line, and, let C be the Point given.



With any Radius, on the given Point C, deſcribe a Semicircle, cuting the given Line in two Points, A and B; i.e. make CB equal CA.

Then, with any Radius, greater than AC, on A and B, deſcribe two Arks interſecting at D, and draw CD.

The Line CD will be perpendicular to AB. Q. E. F.

For, draw DA and DB.

Dem. Then, the Triangles ADC, CDB, are congruous.

Wherefore, the Angles are reſpectively equal, and ACD is equal DCB. - - - - - P. 7. 1.

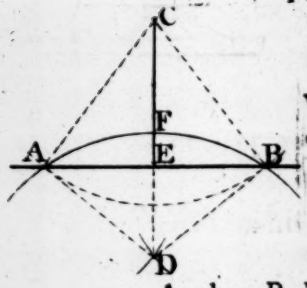
Therefore, CD is perpendicular to AB. - Def. 10.

After the ſame manner, a Perpendicular may be erected on the Ark of a Circle, AEB, at the point E.

PROBLEM VII. 12. I. Euclid.

To draw a Perpendicular to a Right Line, from a given Point which is out of the Line.

Let C be the given Point, from which it is required to draw a Perpendicular to AB.



On C, the Point given, deſcribe an Ark, with any Radius greater than CE, the neareſt diſtance to the given Line, cuting it in A and B. - - - - - - - Poſt. 4.

With the ſame or any other Radius, on A and B, deſcribe two Arks, interſecting at D.

Apply a Ruler to the Points C and D, and draw CE;

Then will CE be perpendicular to AB. Q. E. F.

For, draw AC and AD, BC and BD.

DEM. The Triangles ACD, DCB are congruous. Con.

Wherefore, the Angle ACE is equal to ECB.— P. 7. 1.

Conſ. the Triangles ACE, ECB are alſo equal.— 8. 1.

And the Angles AEC, CEB, being equal, are conſequently Right Angles. - - - - C. 2. 1. 1.

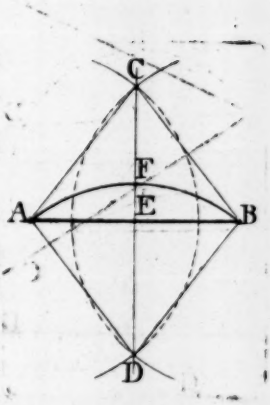
Therefore, CE is perpendicular to AB. - - Def. 10.

In the ſame manner, a Perpendicular may be drawn to the Ark of a Circle, AFB.

PRO-

PROBLEM VIII. 10. I. Euclid.

To biſect a given Right Line, AB.



With any Radius, at pleaſure (provided it be more than half the given Line) on each extreme, A and B, deſcribe two Arks, cuting each other, at C and D.

Draw the Right Line CD; which will biſect, or divide into two equal Parts, the given Line AB, in the Point of their common interſection E. Q. E. F.

Draw AC and AD, BC and BD.

DEM. Then, ACBD is a Parallelogram; by Con. P. 15. 1.
And, the two Diameters AB and CD biſect each other, in its Center, E. - - - - - P. 16. 1.

This Problem may be deduced from and proved as the foregoing.

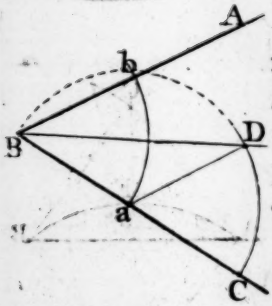
After the ſame manner an Ark of a Circle may be biſected; as AFB in the Point F.

N. B. It is not neceſſary to draw the whole Arks from C to D, but only the Interſections at C and D; the reſt are uſeleſs.

PRO-

PROBLEM IX. 9. I. Euclid.

To biſect a right-lined Plane Angle. ABC.

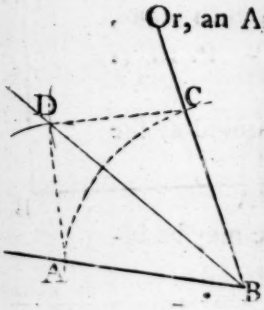


On the Vertex, B, of the given Angle, deſcribe an Ark, ab, at diſcretion, cuting both Sides, in a and b; and, on the interſection a, with the ſame Radius, draw the Ark CD.
Make CD equal ab (Pr. 4.) and draw BD.
Then will BD biſect the given Angle ABC.

Compleat the Semicircle CDAB, and draw aD.

DEM. Then DaC is an Angle at the Center, and DBC is at the Circumference of that Circle.
Th. DaC, (eq. ABC, Con.) is equal twice DBC. P. 9. 3.
And conſequently, ABC is double of DBC. Q. E. D.

Or, an Angle may be readily biſected in this manner.



On the Vertex B deſcribe an Ark, AC, as before.
On the two Interſections, A and C, with the ſame, or any other Radius, deſcribe two Arks, cuting each other, at D.
Draw BD; which biſects the Angle ABC.

Draw DA and DC.

DEM. Then, the Triangles ADB and DBC are congruous, by Conſtruction.
Therefore, the Angle ABD is equal DBC. - P. 7. 1.

Angles may be thus divided into four, eight, or ſixteen equal Parts, by biſecting again and again; but there is no geometrical method, by which, Angles, or curved Lines, may be divided into any equal Parts, at pleaſure, as a Right Line may be divided; otherwiſe than by dividing the Ark with Compaſſes.

APPL.

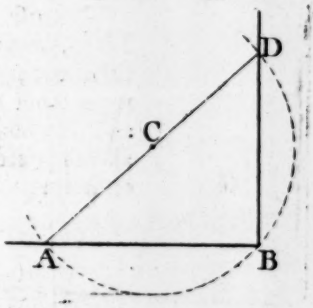
APPL. By this Problem, Carpenters and Joiners, &c. find, what they call, the Mitre of any Angle (whether it be Right, Acute or Obtuſe) with eaſe and expedition.

In returning, or breaking Mouldings at an Angle, whether external or internal, the Angle biſected is the Mitre; in which the Mouldings will exactly fit each other.

PROBLEM X.

To make a Right Angle; or, to draw a Perpendicular at the extreme Point of a Right Line.

It is required to make a Right Angle, at the extreme, B, of the Right Line AB.



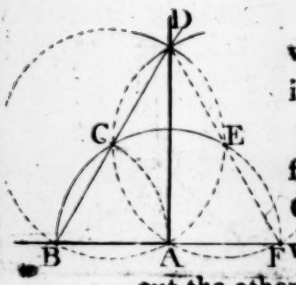
Set one point of the Compaſſes in the point B; and, with any Radius, at diſcretion, fix the other Point, at pleaſure, at C (on that ſide, you require the Perpendicular) and draw the Ark ABD, through the Point B, and interſecting the Line AB in A.
Draw a Right Line through the Interſection A and the Center C, cuting the Ark on the oppoſite Side, at D.
Draw BD, which will be perpendicular to AB. Q. E. F.

DEM. For the Angle ABD, being in a Semicircle, is a Right Angle. - - - - - P. 12. 3.

Or, it is frequently, and not inelegantly, performed thus.

It is required to draw a Perpendicular, to AB, at the extreme Point A.

Fix



Fix one point of the Compaſſes at A, and with any Radius, deſcribe the Ark BCE, cuting AB in B.

On B, deſcribe the Ark AC, cuting the former in C; on which Center deſcribe a Circle, or an Ark AED only; and on E, where it cuts the Ark BCE, draw ACD, or cut the other Ark, only, at D.

Draw AD, which will be perpendicular to AB. Q. E. F.

Or thus, without the Point E.

Deſcribe the Arks BC and AC, interſecting at C.

Through B and C draw the Right Line BCD, indefinite.

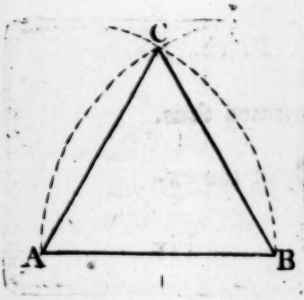
Make CD equal CB, and draw AD; and it is done.

By producing BA to F (making AF equal AB) and FD being drawn, it is demonſtrated as the 6th.

The uſe of Right Angles, perpendicular and parallel Right Lines, are ſo well known, that it would be impertinent to point them out; particularly to thoſe concerned in building, and various other mechanic Arts. They give beauty, ſtrength, utility, and conveniency to a Building and its ſeveral Appurtenances; alſo the execution, of the ſeveral parts thereof, depends on them entirely.

PROBLEM XI. 1. I. Euclid.

To make an Equilateral Triangle, on a Line given, AB.



With the Radius AB, the given Line, and on the extreme Points A and B, deſcribe two, Arks AC and BC, interſecting at C.

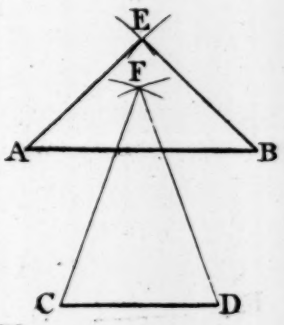
Draw AC and BC; and it is done. Q. E. F.

This needs no Demonſtration; for it is evident, that the three Sides are all equal, ſeeing, they are all Radii of equal Circles.

PRO-

PROBLEM XII.

To make an Iſoſceles Triangle.



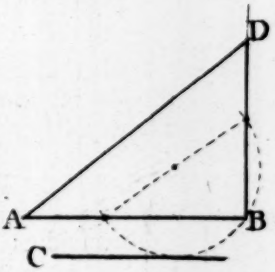
AB and CD are two given Lines; of which, let AB be the Baſe.

Take the Line CD in your Compaſſes, and with that Radius, on the extremes of the other Line, A and B, deſcribe two Arks, interſecting at E; draw AE and EB; and it is done. Q.E.F.

If the other Line, CD, had been required for the Baſe, the operation would be the ſame; taking AB for Radius, & drawing the Arks, interſecting F; draw CF & FD.

PROBLEM XIII.

To make a Right-angled Triangle, having two Lines given for the Sides containing the Right Angle.



Let AB and C be the two given Lines.

On either Line, as AB, and, on either extreme, as B, make a Right Angle ABD.—10. Make BD equal to the other given Line, C,-2. and draw the Hypothenuſe AD; - - Poſt. 2. ABD is the Triangle required. Q E. F.

The uſe and application of Triangles, in general, are almoſt univerſal, in mathematical Sciences. The Iſoſceles and Right-angled are particularly uſeful in Perſpective.

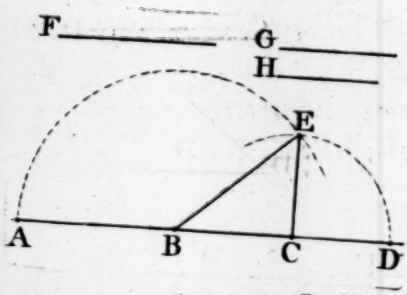
N. B. The Properties of the Right-angled Triangle are fully explained in the ſeventh Propoſition of the ſixth Book of Elements.

PRO-

PROBLEM XIV. 22. I. Euclid.

To make a Triangle, of three unequal Lines given; any two of which muſt be greater than the other.

F, G, and H, are the three given Lines.



Let them be placed at the ends of each other in one Right Line, in what order you pleaſe, i. e. having drawn the Right Line A D indefinite, make the Segments, AB, BC, and CD, reſpectively equal to the given Lines, F, G, H.

With the Radius AB, on B, deſcribe a Circle, or the Ark AE only; and on C, deſcribe the Ark DE, with the Radius CD, cuting the other Ark in E; draw BE and EC, and it is done. Q. E. F.

This Problem and the two laſt need no Demonſtration; every thing being as required, by Conſtruction; agreeable to Def. 26. 28. & 30.

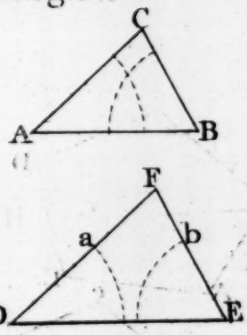
N. B. The Triangle BEC would be the ſame, if either of the other two Lines, F or H, had been made the Baſe, by placing it in the middle; only the poſition of the Triangle would be varied. This follows from Prop. 7. 1. of Elements.

2. There is no neceſſity for placing them, at all, in this manner; only, with the different Radii of two of the Lines, draw Arks, on the extremes of the other Line, cuting each other; taking any one for the Baſe.

PRO-

PROBLEM XV.

To make a Triangle, ſimilar to another Triangle*.



Let ABC be the given Triangle, and DE a Line given, on which to conſtruct a Triangle; ſimilar, and in a ſimilar Poſition, i. e. alike ſituated, to ABC.

At the Point D, of the given Line, make an Angle EDa, equal to CAB, of the given Triangle, by - - - - - - Pr. 4.
And, at the Point E, make the Angle DEb, equal to ABC.
Produce Da and Eb, meeting in F. - - Poſt. 3.
The Triangle DEF will be ſimilar to ABC. Q. E. F.

DEM. For, the three Angles are reſpectively equal. C. 5. 10. 1.
Therefore, the Sides are proportional. - - P. 4. 6.
Conſequently, the Triangles are ſimilar. - - Def. 1. 6.

APPL. This Problem is of great uſe in various Profeſſions. For, by it we learn to take Altitudes and Diſtances, though ever ſo inacceſſible; or, the Surveyor takes his Bearings, and lays down a Plan of the Ground he ſurveys; by it, the Mariner plans the Courſe in which the Ship ploughs the Ocean; and the Mechanic plans the Ground, on which he intends to build, &c.; in ſhort, it is almoſt of univerſal Uſe; which, to enumerate, is not neceſſary in this place.

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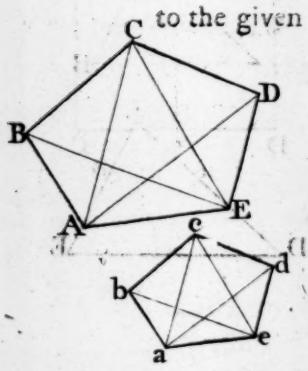
* For ſimilar Figures, ſee Def. 1. of the 6th Book of Elements.

PROBLEM XVI. 18. VI. Euclid.

To make a Figure, ſimilar to any given right-lined Figure.

Let ABCDE be the Figure given, and a e a Line given, on which to conſtruct a Figure, ſimilar, and alike ſituated to the given Figure.



Draw the Diagonals AC, AD, BE, and CE.

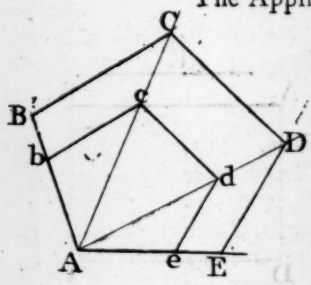
On the given Line, a e, make the Triangle abe, ſimilar to the Triangle ABE, in the original Figure, by the foregoing. - - Pr. 15.

alſo, make ace ſimilar to ACE, & ade to ADE

Join bc & cd, which compleats the Figure.

abcde is ſimilar to ABCDE. Q. E. F.

This method is deduced from the foregoing Problem; for, all right-lined Figures are compoſed of, or may be reduced into right-lined Triangles. It is demonſtrable from 4. and 13. of 6. El. The Application of this Problem is evident.



It may be thus performed, when required bigger.

On AB, the given Line, deſcribe a Pentagon, Abcde, congruous to the given one, by reducing the given Figure into Triangles (as above) and making Abc, Acd, and Ade, reſpectively equal to them. - - - - - Pr. 14.

Produce AE, and the Diagonals, Ac, Ad, indefinite.

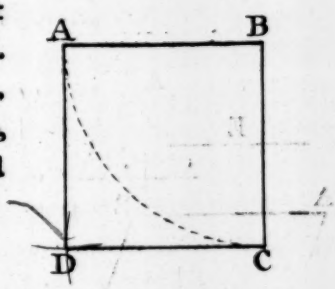
Draw BC, CD, and DE, parallel to the Sides bc, cd, and de, reſpectively; cuting the Diagonals and Side AE, in the Points C, D, and E. ABCDE is ſimilar to Abcde.

After the ſame manner, any Poligon may be increaſed or diminiſhed in any Proportion; demonſtrable by 2. 6. El.

PRO-

PROBLEM XVII. 46. I. Euclid.

To make a Square on a given Line, AB.

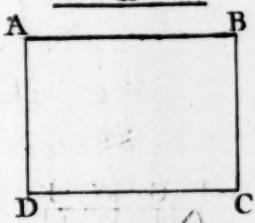


On either extreme of AB, make a Right Angle; as ABC. - - - - Pr. 10.
Make BC equal AB. - - - Pr. 2.
Then, with the Radius AB, on A and C, deſcribe two Arks, interſecting at D, and draw AD and DC.
The Quadrilateral ABCD is a Square.

DEM. For it is equilateral, by Conſtruction; conſequently it is a Parallelogram. - - - P. 15. 1.
Wh. the Angle D = B; conſ. D is a Right Angle.-10. 1.
But, the Angles A and C are alſo equal; conſ. Right.
Therefore, ABCD is a Square. - - - Def. 35.

PROBLEM XVIII.

To make a Rectangle; two Lines is given; AB & E.



Make ABC a Right Angle. - - - Pr. 10.
Make BC equal to the Line E; - - - - 2.
Draw AD parallel to BC, and CD to AB.--5.
ABCD is the Rectangle required.

DEM. For, it is a Parallelogram, - - Con.
Wh. the Angle D is equal B, and A equal C. - P. 15. 1.
But, the Angle B is a Right one. - - - Con.
wherefore they are all Right Angles; - - Th. 1. 10. 1.
and therefore it is a Rectangle. - - - Def. 34.

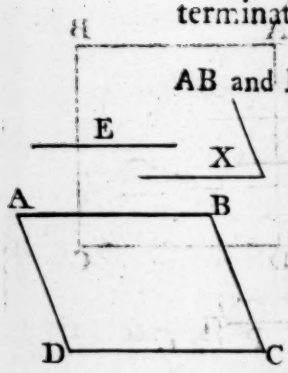
The Square and Rectangle are of great uſe in the mechanic Arts; as moſt regular Figures are Rectangles.

In Menſuration, the Area of every Figure is reduced to a ſtandard meaſure; by the Square, or other Rectangle.

PRO-

PROBLEM XIX.

To conſtruct a Parallelogram under a given Angle, and having its Sides equal to given Lines, of determinate length.



AB and E are the given Lines, and X the given Angle.

At either extreme, of either Line (as A) make an Angle, BAD equal to the given Angle. Make AD equal to E, the other given Line. Through the Points D and B, draw DC parallel to AB, and BC to AD, meeting in C.-Pr. 5. Then is ABCD the Parallelogram required.

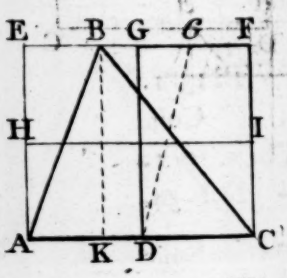
This needs no Demonſtration, every thing being as required by Conſtruction.

The oppoſite Sides and Angles are equal, by - - P. 15. 1.

APPL. By this Problem is delineated Plans, &c. of four ſided Objects which are not right-angled, but, whoſe oppoſite Sides are equal. Any one Angle, being taken, determines all the reſt.

PROBLEM XX. 42. I. Euclid.

To make a Rectangle, or any other angled Parallelogram, equal to a given Triangle.



ABC is the given Triangle.

Biſect any Side, as AC in the Point D.—8. Draw EF, through the oppoſite Angle, B, parallel to AC. - - - - Pr. 5. Draw DG, from the point of biſection, perpendicular to the Baſe AC, cuting EF in G. Laſtly, draw CF parallel to DG.

The Rect. DGFC is equal to the Triangle ABC. Q. E F.

Or,

Or, if the Rectangle AHIC be conſtructed on the whole Baſe, AC, and half the perpendicular Altitude, BK, it will be equal to the Triangle ABC.

DEM. For, it is half the Rectangle AEFC, which is double the Area of the Triangle ABC. - - 17. 1.

N. B. If any other angled Parallelogram was required, you muſt proceed as in Prob. 19. making the Angle CDG or DCF equal to the Angle given; ſtill keeping the ſame Altitude; i. e. between the ſame Parallels, AC and EF, and the ſame Baſe DC or AD.

SCHOL. *From hence, and from the 17th Propoſition of the firſt Book of Elements, the whole Theory of Menſuration is deduced; as all Figures whatever, except Parallelograms, are reſolved into Triangles in Menſuration.*

From the 17th of the firſt Book we learn, that every Triangle is equal to half a Parallelogram of the ſame Baſe and Height; conſequently, the Rectangle DGFC *is equal to the Triangle* ABC, *which is on half its Baſe,* AC, *and the ſame height,* GD, *equal* BK.

COR. Hence, the Rule for meaſuring a Triangle is to multiply the Perpendicular, BK (equal GD) by half the Baſe, AC (equal DC) or the whole Baſe, AC, by half the Perpendicular BK (equal AH); the firſt gives the Rectangle DGFC, the other is the Rectangle AHIC; either of which is equal to the Triangle ABC.

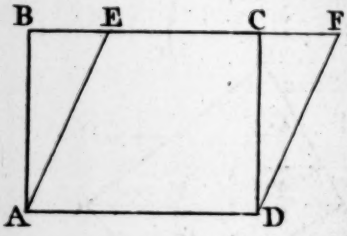
For, if the whole Baſe AC be multiplied by the Perpendicular BK, it gives the Area of the Rectangle AEFC which is double of the Triangle (17. 1,) conſequently, half that Sum is the Area of the Triangle ABC; and alſo of the Rectangle DGFC or AHIC; which is, therefore, equal to the Triangle given.

PRO-

PROBLEM XXI.

To make a Parallelogram, under any given Angle, equal to a Rectangle, and having the ſame Baſe.

ABCD is the given Rectangle.



Make the Angle DAE equal to the given Angle, cuting BC in E, by - - - Pr. 4.
Produce BC; and draw DF parallel to AE.-5.
The Parallelogram AEFD is equal to the given Rectangle ABCD. - - - - P. 18. 1.
Or, it may be thus demonſtrated.

DEM. AB is equal DC (15. 1.) and BE is equal CF. Ax. 7. for, EF (eq. AD) is equal to BC; and EC is common. And, the Angle ABE is equal to DCF. - - P. 4. 1. conſ. the Triangle AEB is equal to DFC. - - 8. 1. Wherefore, if from the Rectangle ABCD there be taken away the Triangle AEB, and an equal Triangle DFC be added, the Par. AEFD is equal to the Rect. ABCD.

COR. 1. Hence it is evident, that two Spaces may have the ſame Area, yet differ greatly in compaſs or circuit.

For, the Sides AE and DF, of the Parallelogram AEFD, are greater than AB and DC of the Rectangle ABCD. 12. 1. But, EF is equal BC (Ax. 3.) for each is equal AD.—15. 1. Therefore, the Circuit of the Par. AEFD is greater than that of ABCD; and they contain equal Areas. - - 18. 1.

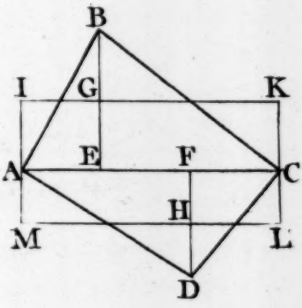
COR. 2. From hence is deduced the general Rule for meaſuring all Parallelograms; which is, to multiply the Baſe, i. e. any Side (as AD) by its perpendicular height, (CD)

For it gives the Area of the Rectangle ABCD, and conſequently of the Par. AEFD, which is equal to the Rectangle.

PRO-

PROBLEM XXII.

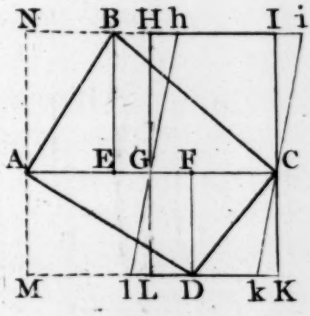
To make a Rectangle equal to a given Trapezium; ABCD.



Draw either Diagonal, as AC; to which, draw the Perpendiculars BE and DF, from the oppoſite Angles. - - - Pr. 7.
Biſect the Perpendiculars, BE and FD, in G and H. - - - - - Pr. 8.
Through the Points G and H, draw IK and LM parallel to AC. - - - Pr. 5.
and through A and C, draw IM and KL, perpendicular to AC. - - - Pr. 10.
The Rectangle IKLM is equal to the Trapezium ABCD.

DEM. For, the Rect. AIKC is equal to the Triangle ABC, and the Rect. ACLM is equal to the Triangle ACD. 20. Conſ. the Rect. IKLM is equal to the Trap. ABCD, Ax. 3.

Otherwiſe,



Having drawn a Diagonal and Perpendiculars, as before; biſect the Diagonal, in the Point G. - - - - Pr. 8.
Draw BI and MK through the Angles B and D, parallel to the Diagonal (AC), and, through G and C, draw HL and IK, parallel to BE and FD, cuting BI and MK in the Points H and L, I and K.
Then, the Rect. HIKL is equal to the Trap. ABCD.

DEM. For, the Rect. GHIC is equal to the Triangle ABC. and the Rect. GLKC is equal to the Triangle ACD. 20.

N. B. By this Problem and the laſt may be performed the 45th of Euclid, without the aſſiſtance of the next, which he makes uſe of, and much readier; making the Parallelogram under a given Angle, as h i k l, which is equal to HIKL by the laſt.

COR. Hence we learn, to find the Area of any Quadrilateral whatever; the Rule for which is, to multiply the Diagonal (AC) by half the Sum of the two Perpendiculars (Fig. 1.) or, the Sum of the two Perpendiculars (BE and FD) by half the Diagonal (GC, Fig. 2.) or, if we multiply the whole Diagonal by the Sum of the two Perpendiculars, they will give the Area of the Rectangle MNIK; half the Sum, of which, is the Area of the Trapezium ABCD.

It may appear ſtrange, to thoſe who have not conſidered it, that the Area of any Figure ſhould be obtained without meaſuring its Sides; which are of no uſe in this operation.

But, having well digeſted what has been advanced, they will find, that the whole buſineſs of Menſuration is to find a Rectangle equal to any Figure; for (as I ſhall make appear hereafter) the multiplication of any two Numbers, applied to meaſure, denotes a Rectangle of ſuch Dimenſions.

Now, ſince it ſeldom happens, that a Trapezium has either Right Angles or parallel Sides, it is plain, that they cannot be of any uſe towards obtaining its Area; whereas, the Diagonals and Perpendiculars are at Right Angles with each other.

A Diagonal divides any Quadrilateral into two Triangles; and every Triangle is equal to half a Parallelogram having the ſame or an equal Baſe, and the ſame Altitude. (Prop. 17. 1.)

Hence, it is eaſy to account for the Rules given for meaſuring a Trapezium, as two Triangles having a common Baſe; which is a Diagonal of the Trapezium.

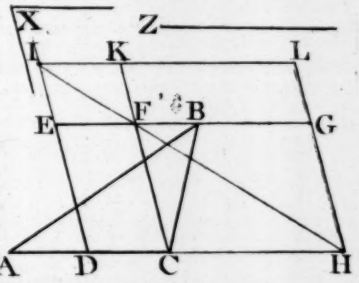
For, every Trapezium is equal to half a Parallelogram which circumſcribes it, having two Sides parallel to either Diagonal; and all Parallelograms having the ſame Baſe and Altitude are equal (18. 1.) conſequently, they are equal to a Rectangle of thoſe Dimenſions.

PRO-

PROBLEM XXIII. 44. I. Euclid.

To make a Parallelogram, equal to a Triangle, having an Angle equal to a given one, and a Side equal to a given Right Line.

ABC is the given Triangle; X the Angle, and Z the given Line.



Make the Parallelogram DEFC, equal to the Triangle ABC, whose Angle, at D or C, is equal to the given Angle, X.—Pr. 20. Produce DE indefinite; also, produce AC; and make CH equal to the given Line, Z;-2. From the Point H, and through the Angle F, of the Parallelogram DEFC, draw HI, cuting DE, produced, in I.

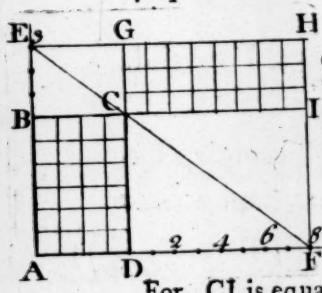
EI, is the other Side of the Parallelogram, fought; which may be compleated by Prob. 19.

Draw IL parallel to AH, and HL parallel to DI meeting at L; produce EF to G, and CF to K.

DEM. The Par. KLGF (having its Angles, at K and G, equal to the given Angle X, and a Side, FG, equal to a given Line, Z) is equal to the Par. DEFC,—P. 19. 1. which, is equal to the Triangle ABC. - - Pr. 10.

SCHOL. *This is, properly, geometrical Division.*

Let the Rectangle ABCD be any Quantity given, to be divided; whoſe Area is 24; the Side AB being 6, and AD 4 equal Parts; it is required to be divided by 3.



Produce any Side, as AB; and make BE equal 3 (half AB) alſo produce AD indefinite. Through E and the Angle C, draw EC, and produce it, cuting AD produced, in F; DF will be equal 8 of the diviſions, or twice AD, which is the Quotient ſought.

Compleat the Rect. AEHF; produce BC to I, and DC to G; the Rect. CH = AC.

For, CI is equal DF, i. e. equal 8; and, CG is equal BE equal 3; 8 multiplied by 3, is equal 24, the Area of CH; which is equal to the Dividend, or the Rectangle ABCD.

If the Diviſor had been 8, equal DF, the Quotient would have been 3, equal BE.

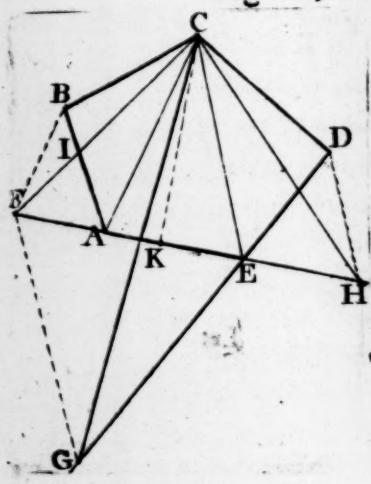
If a fractional Number was given for the Diviſor, the Quotient would, moſt probably, be fractional likewiſe; for the Rectangle under the Diviſor and the Quotient will always be equal to the given Rectangle, which is the Dividend.

As in Diviſion, the Quotient multiplied by the Diviſor is (when there is no remainder) equal to the Dividend; which proves the work to be true.

N. B. It is equal, which way the given Rectangle is ſituated, in this operation; or which Side is produced for the Diviſor; the Quotient, will always be the ſame.

PROBLEM XXIV.

To make a Triangle equal to any given Right-lined Figure; ABCDE.



Draw the Diagonals AC and CE; and produce EA, indefinite. - - - Poſt. 3. From the Angle B, and parallel to AC (the adjacent Diagonal) draw BF; cuting EA, produced in F; and draw FC.

Again. Produce the adjoining Side DE, indefinite; and, parallel to the other Diagonal CE, draw FG; cuting DE produced, in G; draw GC, and it is done. Q. E. F.

The

The Triangle GCD, is equal to the given Pentagon ABCDE.

DEM. 1ſt; the Trap. FCDE is equal to the Pent. ABCDE.
For, ſince FB is parallel to AC, the Triangle AFC is equal to ABC (18. 1.) and AIC is common to both; wherefore, AFI is equal to IBC. - - Ax. 7. 1.
Then, if from the given Pentagon, ABCDE, there be taken away the Triangle IBC, and its equal FAI be added; the Trapezium FCDE is equal to the given Pentagon. - - - - - - Ax. 7. & 6.
But, the Trap. FCDE is equal to the Triangle GCD.
For, becauſe FG is parallel to CE, the Triangle EGC is equal to EFC. - - - - - - P. 18. 1.
conſ. EGC added to ECD = EFC + ECD.— Ax. 6. 1.
i. e. the Triangle GCD is equal to the Trap. FCDE.
But, the Trap. FCDE= the Pent. ABCDE.-proved above
Th. the Triangle GCD is equal to the given Pentagon.

Or, if the Side AE had been produced both ways, and DH drawn parallel to CE, cuting AE in H, and if HC be drawn; the Triangle FCH is alſo equal to the given Pentagon, equal to the Triangle GCD.
For, the Triangle CHE is equal to CDE. - - 18. 1.

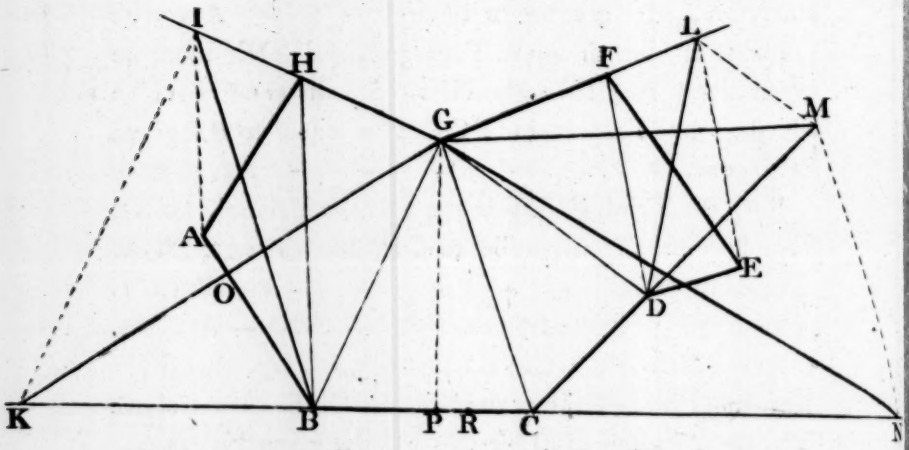
SCHOL. *By which means, a Triangle may be readily conſtructed, whoſe Altitude ſhall be equal to the Altitude of the Poligon; or, to a Perpendicular, from any Angle to any Side; as* CD *or* CK. *Or, any one Angle, as* D, *and an adjoining Side,* CD, *of the Poligon, may be retained in the Triangle.*

I have been more particular in the Demonſtration of this Problem than in any of the former; becauſe, it may, to ſome, appear, at firſt ſight, rather intricate; which it certainly is not, being properly analized.
From hence, and from Prob. 20. a Rectangle may be found or conſtructed equal to any given right-lined Figure; and, by Prob. 25. the Side of a Square, equal to a given Rectangle, is readily obtained.

But

But, as the operation, in a more complex Figure, may not be readily performed, without an Example, I ſhall briefly go through the whole proceſs; and alſo, ſhew how much it is preferable to the 14th Prop. of the 2d Book of Euclid; viz. To find the Side of a Square, which is equal to any given right-lined Figure.

Let A B C D E F G H be the given Figure.



Firſt, draw the Diagonals B H, B G, G C, &c.
Produce the Side G H, and, draw A I, parallel to the Diagonal B H; cuting G H produced, at I, and draw B I.

Now, ſince A I is par. to B H, the Tri. B H, = B A H. 18. 1. wherefore, the Tri. B I G, is equal to the Trap. B A H G.

Next, produce the Side B C, both ways, indefinite, for the Baſe of the Triangle ſought.

From the new-acquired Angle I, draw I K, par. to the Diagonal B G; cuting C B produced, at K, and draw G K. Then, the Tri. B K G = B I G, having the ſame Baſe B G. 18. 1.

But the Tri. B I G is equal to the Trap. B A H G;
wh. the Tri. B K G, is equal to the Trap. B A H G.---Ax. 3.
Conſ. the Triangle B K O, which is added, is equal to the Trap. A H G O, taken away from the original Figure; for, B O G is common to both.

Again,

Again; produce GF; and, from the Angle E, draw EL, parallel to the Diagonal FD; and draw DL.
The Triangle FLD is equal to FED. - - - 18. 1.
The given Figure is now reduced to a Pentagon, KGLDC.

Then; produce the Side CD; and, from the Angle L, draw LM, parallel to the Diagonal GD; cuting CD produced, in M, and draw GM.
The Triangle GMD is equal to GLD. - - 18. 1.

Laſtly; from the Angle M, and par. to the Diag. GC, draw MN, cuting KC produced, in N, and draw GN.
The Triangle GNC is equal GMC; equal to the Trapezium GLDC; equal GFEDC.

But, the Triangle BKG is equal the Trap. BAHG; wherefore, BKG, added to BGC, added to GNC, is equal to the Triangle KGN. - - - Ax. 2.
Conſ. the Triangle KGN, is equal to the Trap. KGMC; which is equal to the Pentagon KGLDC;
equal to the Hexagon KGFEDC;
equal to the Heptagon BIGFEDC;
equal to the Octagon AHGFEDCB.
Therefore, the Triangle KGN, is equal to the given Figure.

Having thus reduced the given Figure to a Triangle, it is eaſily formed into a Rectangle, by Prob. 20.;
For, a Rectangle on KR, half the Baſe KN, and GP, the perpendicular Altitude of the Triangle, is equal to the Triangle. - - - - Cor. 17. 1.

Or, by Prob. 23. it is eaſily reduced to a Parallelogram under any given Angle and Side.

SCHOL. *Let*

SCHOL. *Let any one compare this, with the* 14*th Propoſition in the* 2*d Book of Euclid*; *that is, let him go through the operation, both ways, in a Figure of as many Sides*; *I am confident to which he will give the preference. To ſay nothing of the inaccuracy of the other, this may be done in leſs than a fourth part of the Time. Each Triangle (of which there are ſix, in this Figure) goes through two operations,* viz. *20th and* 23*d of this*; *and are added, ſeparately into one Sum, or Rectangle. Whereas, by this method, the Figure is reduced one Side at every operation*; *from a Hexagon to a Pentagon, from a Pentagon to a Trapezium, from a Trapezium to a Triangle*; *each, being equal to the original Figure: and, from a Triangle to a Rectangle, or other Parallelogram under any Side or Angle*; *by the* 23*d.*

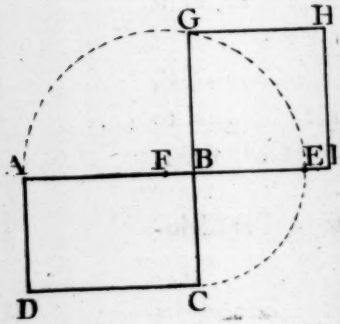
APPL. This Problem, may be applicable in ſeveral Caſes, in ſurveying, &c. with the greateſt accuracy. e. g.

If you would exchange an irregular piece of Ground in one Place, for the ſame Quantity in another, ſuppoſe to build on; which, by reaſon of the contiguous Buildings, is confined to a certain Angle, which is the given Angle; the length in Front may be conſidered as the given Side; the Queſtion is, what Depth of Ground from the Front is required, to be equal in its Area to the other.

Having firſt reduced the original given Figure to a Triangle, by this Problem, it is then convertible into a Parallelogram under any Side and Angle, by the 23d; or, into a Square by the following.

PROBLEM XXV.

To make a Square, equal to a Rectangle. ABCD.



Produce any Side, as AB, of the given Rectangle, until it be equal to the adjoining Side, BC; i. e. make AE equal to AB and BC, in a Right Line.

Biſect AE, in F; on which, with the Radius AF, equal FE, deſcribe a Semicircle.

Produce CB, till it cuts the Arch, at G; or, at the point B, draw BG perpen. to AE.

Deſcribe the Square BGHI on the Line BG.

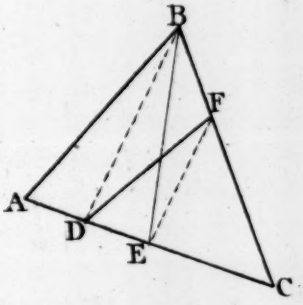
DEM. It

DEM. It will be equal to the Rectangle ABCD. - - 14. 3.
Alſo, BG is a mean Proportional between AB and BE.
Conſ. the Rectangle under AB and BE (i. e. DB) is equal to the ſquare of BG (i. e. BH) by—Cor. to 9. 6.

N. B. In the performance of this Problem it is not neceſſary to conſtruct the Rectangle; but only, to draw a Right Line (AE) in which, take AB and BE equal to the meaſures of the two Sides, and proceed as above.

PROBLEM XXVI.

To divide a Triangle into two equal parts, by a Right Line drawn from any Point given, in a Side.



Let D be the given Point, in the Side AC, of the Triangle ABC.

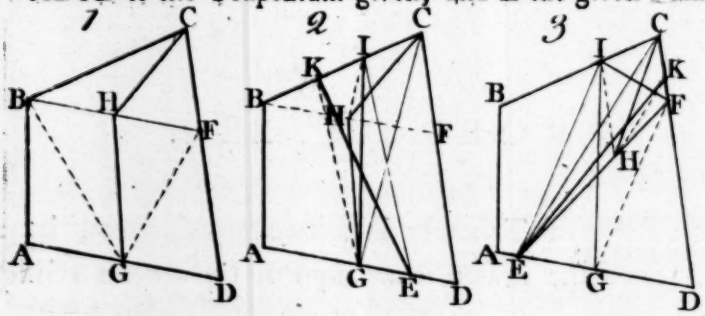
Biſect the Side AC, in E, and draw EB, to the Vertex B.
Draw DB, and EF parallel to it, cuting BC in F, and draw DF.
The Trapezium ABFD, is equal to the Triangle DFC.

DEM. Becauſe AE=EC, the Tri. ABE=EBC.— 38. 1.
But, EF is par. to DB; wh. the Tri. BFD=DBE--ſame. conſ. BFD+ABD is equal to DBE+ABD. - - Ax. 6.
And, the Triangle ABE is equal to EBC. - - - above.
wh. the Trap. ABFD, is equal to the Tri. ABE, eq. EBC; and conſequently, to the Triangle DFC.
Therefore, the Right Line DF divides the Triangle ABC into two equal Parts. Q. E. F.

PROBLEM XXVII.

To divide a Trapezium into two equal Parts, by a Right Line, drawn from a Point given in any Side.

ABCD is the Trapezium given, and E the given Point.



Through the Angle B, draw BF, parallel to AD. — 5. Biſect AD and BF, in G and H; and draw GH and HC. Then the Pentagon ABCHG is equal to the Trap. GHCD.

DEM. For (having drawn BG and GF) the Triangle ABG is equal to GFD, the Triangle BGH is equal to HGF, and BCH is equal to HCF. - - - - P. 18. 1.

2dly. In Fig. 2. let ABCHG be equal GHCD, as before. Draw GC, and HI parallel to GC, cuting BC in I, and draw GI.

The Right Line GI biſects the given Trapezium.

DEM. For, becauſe HI is parallel to GC, the Triangle GIC is equal to GHC. - - - - - P. 18. 1.

Wherefore, GIC added to GCD=GHC+GCD. Ax. 6.

Again. Draw EI, and GK parallel to EI, cuting BC in K, and draw EK.

I ſay, that EK divides the Trapezium ABCD equally.

DEM.

DEM. For firſt, the Trap. GHCD was proved half ABCD;
ſecondly; GICD was proved equal to GHCD;
and, EKCD is equal to GICD.
For the Triangle EKI is equal to EGI. - - P. 18. 1.
Therefore, the Trap. ABKE is equal to EKCD.

Now, if E had been given near the Angle A, at the leſſer end of the Trapezium, the operation would require more labour.

Let Fig. 3. be ſuppoſed the ſame Figure divided, by the Line GI, into two equal parts, from the middle Point G (as by the ſecond operation) which is neceſſary to be firſt done, in all caſes.

Join the given Point, E, and I, as before; draw GF, parallel to EI, cuting CD in F, and draw EF and IF.
Then, the Pent. ABIFE is equal to the Tri. EFD + FIC.

For, becauſe GF is parallel to EI, the Tri. IFE = EGI.

But, although the Pentagon ABIFE is equal to half the Trap ABCD, it is not equally divided by one Right Line. Therefore, draw IH parallel to CD, cuting EF in H, and draw HC.

Then, becauſe IH is parallel to CF, the Triangle ICH is equal to IFH. - - - - - - - - 18. 1.
Wh. the Pent. ABCHE = ABIFE = the Trap. EHCD.

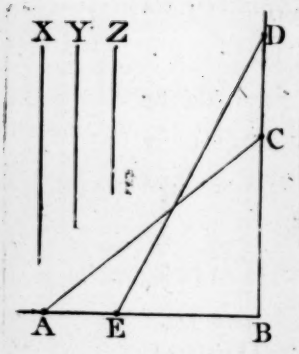
Laſtly; join EC, and draw HK parallel to EC, cuting CF in K; and draw EK; which Line divides the Trapezium ABCD into two equal parts, as required.

For, ſince HK is parallel to EC the Tri. EKC = EHC; wherefore, EKC added to ECBA = EHC added to ECBA.
But, ABCHE was equal to ABIFE, equal to ABIG; which, was equal to half the Trapezium ABCD.

PROBLEM XXVIII.

To find the Side of a Square, equal to any Number of Squares.

Let X, Y and Z be the Sides of three given Squares.



It is required to find the proportion of a Line, on which if a Square be conſtructed, it ſhall be equal in Area to all the three.

Make a Right Angle ABD. - - Pr. 10.
From the Angle B, take BA and BC, equal to any two of the given Lines, reſpectively, as X and Y, and join AC.
The Square of AC is equal to the two Squares of AB and BC, or X and Y. - - - P. 20. 1.

Again. Make BD equal AC, and BE equal Z;
Draw ED, which is the Line required.

DEM. For, ED ſquare = the two Squares of EB & BD.
But, BD, equal AC, ſquare = AB + BC ſquare.
Wherefore, ED ſquare = AB + BC + EB ſquare.
i. e. ED ſquare = the three Squares, of X, Y and Z.--20. 1.

APPL. By this uſeful Problem, Quantities may be increaſed in any Proportion at pleaſure.

Alſo, by means of this Problem, and Prop. 20. 1. Carpenters form a Right Angle, in framing Timber, &c.

For, having made AB equal 3 feet, and BC equal four; then, if AC meaſures 5 feet ABC is a Right Angle.

N. B. The Numbers 3, 4 and 5 being multiplied, ſeparately, by any one Number, at pleaſure, produce the ſame effect. e. g.
If AB be made 6 or 9 feet, and BC equal 8 or 12 feet, then will AC be equal to 10 or 15 feet, if the Angle ABC be a right one.
For, the Square of 9 is 81; and the Square of 12 is 144, added to 81 is 225; equal to the Square of 15.——20. 1. El.

When

When the Timbers are long, it is neceſſary, in placing them at right angles, to apply greater meaſures; which, will give the Right Angle with greater accuracy.

Cor. Hence, a Perpendicular may be drawn, very readily, at the extremity of a Right Line; by a Scale of equal Parts. As BC, perpendicular to AB. e. g.

Take, by any Scale, 3, 6, 9, or 12 equal Parts, which, ſet off, from B to A; at which point, B, a Perpendicular to AB is required.

Then take, for Radius, 4, 8, 12, or 16 Diviſions, of the ſame Scale; and, ſeting one Point of the Compaſſes in B, make a ſmall Ark, at C.

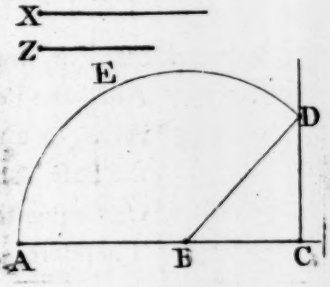
By the ſame Scale, take 5, 10, 15, or 20 equal Parts, and with one point of the Compaſſes, at A, croſs the former Ark, at C, and draw BC; which will be perpendicular to AB.

For, the Square of AC is equal to the two Squares of AB and BC, added together; by 20. 1. El.

PROBLEM XXIX.

The Sides of two Squares being given, to find the Side of a Square which is equal to the difference between them; i. e. by how much the greater exceeds the leſs.

X and Z are the Sides of the given Squares.



Draw AC indefinite; in which, take AB equal X, and BC equal Z. - - - Pr. 3. Make ACD a R. Angle, and draw CD, indef. On B, with the Radius AB, deſcribe the Ark AED, cuting CD at D; CD is the Side of a Square, equal to the difference between the Squares of AB and BC, or X & Z. Draw BD.

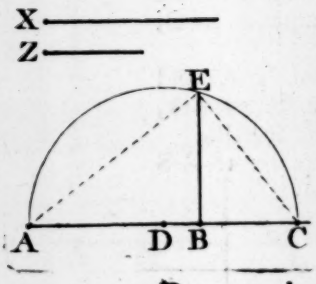
Dem. BD (equal AB, equal X) is the Hypothenuſe of the right-angled Triangle DCB; the ſquare of which, is equal to the ſquare of BC added to the ſquare of CD. Conſequently, BD ſquare (equal X) exceeds BC ſquare (equal Z) by the ſquare of CD; by Prop. 20. 1.

PRO-

PROBLEM XXX. 13. VI. Euclid.

To find a mean Proportional between two given Lines.

X and Z are the two given Lines.



It is required to find a Line, to which, either of the Lines, X or Z, ſhall have the ſame Ratio or Proportion, as that Line has to the other. Or, the Square of which, ſhall be equal to a Rectangle under the two given Lines.

Draw at pleaſure AC; make AB equal to one of the given Lines, as X, and make BC equal to the other, Z.

Biſect AC, in the point D; on which Center, and with the Radius AD, equal DC, deſcribe a Semicircle.

At the Point B, draw a Perpendicular, to AC, cuting the Ark at E; and BE is the Line ſought.

DEM. For, draw AE and EC; AEC is a Right Angle--12.3. And, the Perpendicular, BE, in a right-angled Triangle, AEC, is a mean Proportional, between the Segments of the Baſe, AB, BC, made by the Perpendicular. C.1.7.6.El. Conſequently, AB is to BE, as BE is to BC. — ſame. Therefore, AB×BC = the ſquare of BE.—Cor. to 9. 6.

SCHOL. *This is the very ſame, in the operation, as the 25th; for the Side of a Square is a mean Proportional between the two Sides of a Rectangle having an equal Area.*

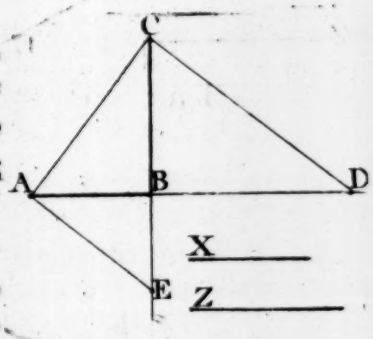
PRO-

PROBLEM XXXI. 11. VI. Euclid.

To find a third Proportional to two given Lines.

X and Z are the given Lines.

It is required to find a third Line, to which the greater, Z, ſhall have the ſame proportion, as the leſſer, X, has to Z. Or, which ſhall have that proportion to the leaſt; as the leaſt to the greateſt.



Make a Right-angled Triangle, ABC, whoſe Catheti or Legs, are equal to the two given Lines, i. e. make AB equal to one (X) and BC equal to the other.---13 Produce AB and CB, indefinite, Make ACD and CAE Right Angles; i. e. draw CD and AE perpendicular to AC, cuting AB and CB, produced, in D and E.

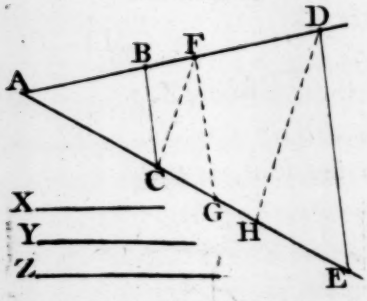
Then will BD be a greater, and BE a leſs, third Proportional, to the two given Lines, X and Z; Q. E. F.

DEM. For, as in the laſt, EB is to AB, as AB to BC; and AB : BC : : BC : BD. - - - - P. 7. 6.
Wherefore, EB, AB, BC, and BD are in continual Proportion. - - - - - - - Def. 7. 5.
Conſequently EB is a leſs, and BD a greater third Proportional, to the two Lines AB and BC, equal X and Z.

PRO-

PROBLEM XXXII. 12. VI. Euclid.

To find a fourth Proportional, to three given, unequal Lines; X, Y and Z.



It is required to find a Line, to which, the third (Z) ſhall have the ſame proportion, as the firſt (X) has to the ſecond (Y). Or, the ſame proportion to the firſt Line as the ſecond has to the third.

Draw two Lines, AD & AE, making any Angle at pleaſure.
Make AB equal X, & AC equal Y; alſo make BD equal Z.
Draw BC, and DE parallel to it; cuting AE in E; and,
CE is a fourth Proportional, greater than Z.
i. e. As AB is to AC, ſo is BD to CE. Q. E. F.

DEM. For, becauſe BC is parallel to DE, the Sides of the Triangle ADE are cut proportionally;
wherefore, AB : AC, as BD : CE. - - P. 2. 6.
But, AB is equal to X, AC is equal Y, and BD equal Z; therefore, as X is to Y, ſo is Z to CE.

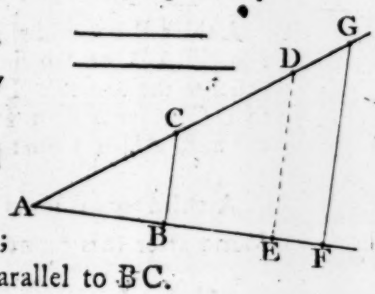
If a leſs Proportional be required; make AG equal to Z, make AF equal Y, and GE equal to X.
Join FG, and draw DE parallel to it.
Then, will FD be the Proportional ſought.

DEM. For, as AG (equal Z) is to AF (equal Y) ſo is GE (equal X) to FD. - - - - by the ſame.

After

After the ſame manner, a third Proportional may be found.

Draw AF and AG, making any Angle, as before.
Let AB & AC be made equal to the given Lines, reſpectively



If the greater Proportion be required, make BF equal to AC, and draw FG, parallel to BC.
The Segment CG is the Proportional ſought.

But if a leſs Proportional is wanted; make CD equal AB, and draw DE parallel to BC.
Then, BE is the Proportional required.

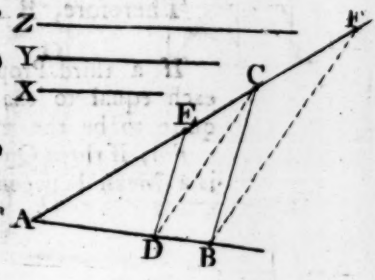
DEM. For, as AB is to AC, ſo is BF to CG;
and, as AC is to AB, ſo is CD to BE. - - 2. 6. El.
But, BF was made equal to AC, and CD to AB.
Conſ. as AB : AC :: AC: CG ; and, AC : CD :: CD : BE.

N. B. A fourth Proportional to three given Right Lines may be required and found in various orders of the given Lines.

Let X, Y, and Z be three given Lines; of which, let X be the leaſt, and Z the greateſt of the three.
By the firſt, it is, as X is to Y, ſo is Z to a fourth;
and by the ſecond, as Z is to Y, ſo is X to a fourth.
But it may be, as Y is to either X or Z, ſo is the other to a fourth.
Alſo, as X is to Z, or Z to X, ſo is Y to a fourth.

It may alſo be obſerved, that it is not neceſſary to draw the Lines, making an Angle, for the operation, longer than the greateſt given Line; except when a Proportional is required greater; for the meaſures may all be ſet off from the Vertex. e. g.
Let AB and AC make any Angle (BAC) at diſcretion.

Make AB equal to Y, and AC equal Z; alſo, make AD equal X.
Join BC; and draw DE, parallel to BC, cuting AC in E.
Then, as AB or Y, is to AC or Z; ſo is AD or X, to AE, a fourth.
But, if DC be joined, and BF drawn, parallel to DC; it is then, as AD or X, is to AC or Z, ſo is AB or Y, to AF, a fourth.



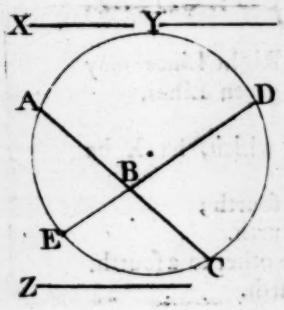
Note. This method, of ſeting off all the meaſures from the Angle, is the moſt eligible, when we are clear in the manner of placing them; being leſs liable to error than when they are ſet forward on a longer Line; on account of the Parallels being nearer together. The Demonſtration is in the 4th of 6th El.

A third Proportional may alſo be found after the ſame manner. For, if AD be the firſt; and, if AB and AC be each made equal to the ſecond; DC being joined, and BF drawn parallel to DC gives AF, a greater third Proportional; or, if AF be the firſt, AD is a leſſer third.

A third or a fourth Proportional may be very elegantly found after this manner.

Let X, Y and Z be three given Lines.

If you require a leſs Proportional, X is the firſt Term taken; if a greater be required, Z muſt be the firſt, contrary to the order, after the former method; nor can the Terms be taken alternately, as in the other.



Draw a Right Line, AC, at pleaſure. Make AB and BC equal, reſpectively, to the firſt and ſecond Terms, X and Y. Through the point B, draw, at pleaſure, DE and make BD equal to the third Term (Z) Deſcribe a Circle through the three Points A, C, and D, cuting DE, at E.—Pr. 40. Then, BE is a fourth Proportional; in the order Z to Y, as X to BE.

Dem. For, the Rectangles under the ſegments of Chord Lines, cuting each other, are equal, i. e. the Rectangle, under AB and BC, is equal to that under DB and BE. Conſ. as DB : AB, or BC : : BC, or AB : BE.- 9. 6. El. Therefore, BE is a fourth Proportional.

If a third Proportional be required, AB and BC muſt be each equal to that Term, of two given Lines, which you require to be the middle Term of the three.

For, if three Quantities are Proportionals, the middle Term is a Mean between the other two.

In

In analogous or equal Proportion of four Quantities, ſince the firſt has the ſame proportion to the ſecond, as the third has to the fourth; and conſequently, the firſt is to the third as the ſecond to fourth; a Rectangle under the two extreme Terms (in equal Ratios of Right Lines) is equal to a Rectangle under the two middle Terms (9. 6. El.); which, in Numbers, is eaſily proved.

Take any four proportional Numbers, either continual; e. g. as 3, 6, 12, and 24; or in equal Ratios, as 3, 5; 9, and 15.

Now, it is plain that the firſt Number, 3, has the ſame proportion to the ſecond, 5, as 9, the third, has to the fourth, 15.

For, if 3 be multiplied three times, it is equal 9; and 5 multiplied three times, is equal 15; conſequently, 3 has the ſame proportion to 9, the third Number, as 5, the ſecond, has to 15, the fourth; and ſo it will ever be, when any two Numbers are multiplied equally.

As 4 is to 6, ſo is 10 to 15; conſequently, 4 : 10 :: 6 : 15.

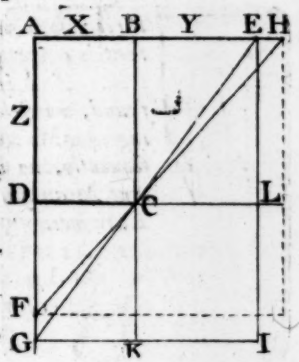
Now, in both theſe Caſes, the two extreme Terms, i. e. the firſt and the laſt, viz. 4 and 15, and the two middle Terms, 6 and 10, remain the ſame; only, the middle Terms have changed places; but, the ſecond multiplied by the third or the third by the ſecond is the ſame thing; and is always equal to the fourth multiplied by the firſt, or the firſt by the fourth.

Wherefore, if four Lines are proportional, as above; a Rectangle under the firſt and the fourth, the two Extremes, is equal to a Rectangle under the two mean or middle Terms; that is, the Rectangles have equal Areas, ſeeing, the Area, of a Rectangle, is produced by the multiplication of one Side by the other.

Hence, a fourth Proportional may very readily and accurately be found; as follows.

Let X, Y and Z be three given Lines.
It is required to find a fourth Proportional, which ſhall have the ſame Ratio or Proportion to Z, as X has to Y.

In this Caſe, X and Z will be the two Extremes; for ſince X is leſs than Y; conſequently, the Proportional required, will be leſs than Z, and is, properly, a Mean.



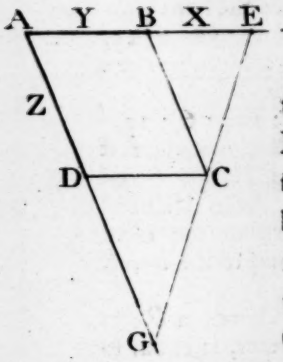
Make a Rectangle, ABCD, under the two given Lines X, and Z.

Produce any two Sides, from the ſame Angle, as AB and AD; on either of which, make BE or DF, equal Y.

Draw EG or FH, through the Angle C, cuting the other Side, produced, in G or H; then is DG, or BH, the Proportional ſought.

For, compleat the Rectangle AEIG.
Produce BC to K, and DC to L.

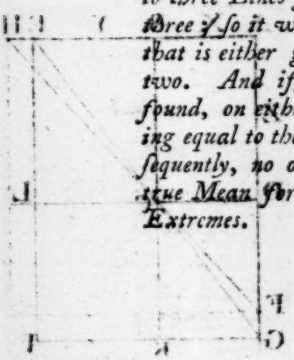
DEM. The Rectangle CI, is under BE (equal Y) and DG, the fourth Proportional required.
AC and CI are Complements of the Par. AEIG. Def. 38.
But, the Comp. in every Parallelogram are equal. - 19. 1.
Th. the Par. or Rect. CI is equal to the Rectangle ABCD.



But, if the Proportional was required to be to Z, as Y to X; then the Rectangle, or any Parallelogram, ABCD, muſt be under the two Lines Y and Z; which will, in this Caſe, be the two Means; and DG the Proportional ſought, is now one of the Extremes; being the greateſt of the four; which, in the former Caſe, was one of the middle Terms.

Thus may a fourth Proportional be found, either greater or leſs than either of the three given Lines, X and Z.
For, if a leſs Proportional was required, which ſhould be an Extreme of the four; the Rectangle or Parallelogram, muſt be made under X and Y.

SCHOL. *When three Lines are given; a fourth Proportional is generally underſtood, to be either greater than the greateſt, or leſs than the leaſt of the three; but if another mean Proportional is required, to three Lines given, as it muſt be between the two Extremes of the three; ſo it will be, either greater or leſs than the middle Line, as that is either greater or leſs than a true Mean, between the other two. And if it be already a true Mean, there can no other be found, on either Side; for, the Square of a mean Proportional, being equal to the Rectangle under the two Extremes* (C. 2. 6.) *conſequently, no other Rectangle but a Square, which ſhall have the true Mean for its Side, can be equal to a Rectangle under the two Extremes.*



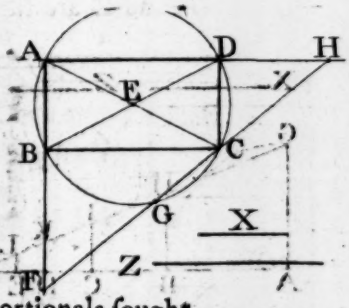
PRO-

PROBLEM XXXIII.

To find two mean Proportionals, to two given Lines.

Let X and Z be the two given Lines.

It is required, to find two other Lines; which not only contain an equal Rectangle, but, are in continual Proportion.



Conſtruct the Rectangle ABCD, on the two given Lines; and on its Center, E, deſcribe a Circle, circumſcribing it.
Produce the two Sides AB & AD, indef.
Apply a Ruler to the Angle C, and move it on that Point, till it makes FC & GH equal; and draw FH through G and C.
Then will BF and DH be the two Proportionals ſought.

For, DC, DH, BF, and BC are in geometrical Progreſſion.
And, a Rectangle under the two Means, DH and BF, is equal to the Rect. ABCD, under the two given Lines, X & Z.

Or, briefly thus.

Make a Right Angle FAH; in which, from the Angle A, make AB & AD reſpectively equal to the given Lines, X & Z.
Join BD; which biſect in E; and, on E, with the Radius EB, deſcribe a Semicircle, BCD.
Make DC equal AB (eq. X) and apply a ſtreight Ruler to the Point C, making EF equal to EH; and it is done. Q.E.F.

N. B. This method, though very ingenious, is not perfectly geometrical; ſeeing, the Points F and H cannot be aſcertained, but by trial; yet, it is the beſt I have met with; for thoſe methods which are performed by an Inſtrument, are not practical otherwiſe, and conſequently cannot be called geometrical.

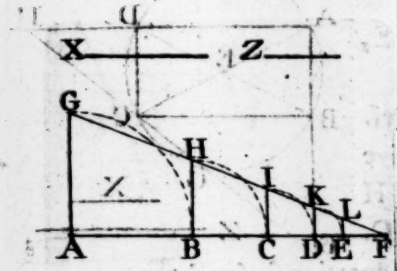
The method for finding a third Proportional (Pr. 31.) exhibits the reverſe of this; for AB and BC, the two given Lines, in that Problem, are two Means, between the two Extremes BE and BD. For, BD : BC :: BC : BA :: BA : BE. - - - - P. 7. 6.

PRO-

PROBLEM XXXIV.

To continue a progreſſive Proportion, between two given Lines, infinitely; and to repreſent the ſum of them all.

X and Z are the two given Lines.



Draw an indefinite Right Line, AF; in which, take AB equal to X, and BC equal Z.

Make BAG a Right Angle; and draw BH parallel to AG.

Make AG equal AB, and BH eq. BC.

Through the Points G and H, draw a Right Line, cuting AC, produced, at F; and AF is the whole Sum of the infinite Proportionals.

Draw CI par. to BH; make CD equal CI, and draw DK.

Make DE equal DK, and draw EL, perp. to AF.

Then is CI a third, DK a fourth, and EL a fifth Proportional; and, after the ſame manner, it may be continued, *ad infinitum.*

For, ſince AG is leſs than AF; ſo BH is leſs than BF; and conſequently, CI, DK and EL, will ſtill have the ſame Ratio to CF, DF and EF; wherefore, the laſt may always be taken from the remainder, and therefore, AF, is equal to the whole ſum of the infinite Proportionals; and AB, BC, CD, &c. or, AG, BH, CI, &c. are in a progreſſive, geometrical Proportion.

The Demonſtration of this Problem and the laſt are given at the end of the ſixth Book.

PRO.

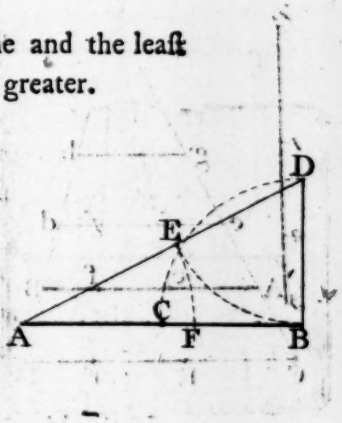
PROBLEM XXXV. 11. II. Euclid.

To divide a Line in extreme and mean Proportion.

AB is the given Line.

It is required to cut it ſo, that the leſſer Segment ſhall have that Proportion to the greater, as the greater Segment has to the whole Line.

Or, the Rectangle under the whole Line and the leaſt Segment ſhall be equal to the Square of the greater.



Biſect AB, in C; draw BD perpendicular to AB, and equal to BC, half the given Line, and draw AD.

On D, with the Radius DB, deſcribe an Ark BE, i. e. make DE equal DB; and, on the point A, deſcribe the Ark EF, cuting AB in F, the Point ſought. Q. E. F.

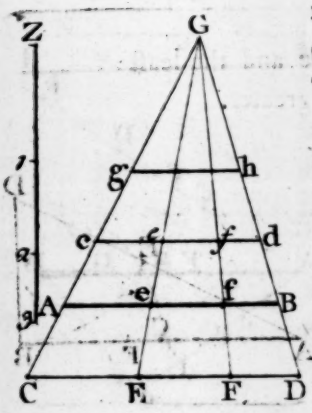
N. B. AF, the greater Segment, is the difference between the leaſt Side, DB, and the Hypothenuſe, AD, of the Right-angled Triangle ABD; conſtructed on the whole given Line, and half the Line, for the Baſe and Perpendicular. - - - - Pr. 13.

This Problem is otherwiſe performed and demonſtrated in Prop. 11. of the ſecond Book of Elements.

PROBLEM XXXVI. 10. VI. Euclid.

To divide a given Line, in any known Proportion.

AB is the given Line; and Z, is a Line divided in the Proportion required, at 1, 2, and 3.



At any diſtance from AB, at diſcretion, draw CD parallel to AB; and make CE, EF, and FD equal, reſpectively, to the Diviſions on Z.

Through the extreme Points of the two Lines, AB and CD, draw CA and DB, produced till they interſect, at G.

Draw EG and FG, cuting AB in e and f.

Then, is the given Line, AB, divided (in e and f) in the ſame Ratio as CD (in E and F) or as the given Line, Z, is divided, in the Points 1, 2, 3. - - - - P. 2. 6. El.

If the Meaſure given had been leſs than the Line given to be divided (as in this Example it is greater) AB would be divided the ſame.

Let cd, leſs than AB, be divided in the known Ratio, *e* & *f*.

Draw, as before, the Lines Ac, and Bd, through their Extremes, meeting at G; and, from G, through the Diviſions *e* and *f*, draw the Lines G*e*, G*f*, cuting the given Line, AB, in the ſame Points e and f. Q. E. F.

The difference, it is evident, is only in the operation, for the effect is the ſame.

In the firſt Caſe, when the Meaſure is greater than the Line to be divided; then, the Vertex G will fall on the oppoſite Side, from CD; but, when it is leſs, as cd, the Vertex, G, will fall on the ſame Side with cd; as is obvious from the Figure.

N. B. Any

N. B. Any other Line drawn between the two Lines GC and GD, as gh, parallel to AB or CD, will be divided in the ſame Ratio, by the Lines GE and GF.

2. The ſame thing may be very readily and accurately done after this manner.

AB is the given Line, to be divided; and HI is a Line divided in the given or known Ratio, in 1 and 2.

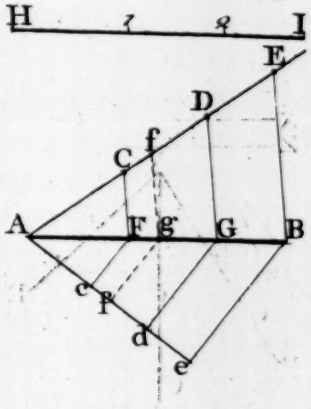
At either extreme of the given Line, draw AE, in any Angle at pleaſure.

An acute Angle, not too ſmall, is beſt.

Make the diviſions at C, D, and E, equal, reſpectively, to the diviſions on HI; according as you require them on AB, begin at either end, H or I.

Join the Point E, and the other extreme, B, of AB; draw DG and CF, parallel to EB, cuting AB in G and F.

So ſhall AB be divided in the ſame Ratio, as HI, in F and G, - - - P. 2. 6.



If the Meaſure given had been leſs than AB, as Ae, divided in c and d, the Diviſions, on AB, would be the ſame.

N. B. If it was required to divide AB into any number of equal Parts; make ſo many equal Diviſions on CD, in the firſt, or on AE, in the laſt method, at pleaſure; and proceed as directed.

AB may be readily biſected by either method; by the laſt, making Af, fE, or fe, two equal diviſions, and drawing fg, as before, parallel to EB or eB.

The firſt method is demonſtrable, from Prop. 6. 6. & Corol. For the Triangles CGe and AGe, alſo EGF and eGf, &c. are ſimilar. Conſ. Ae : ef : : CE : EF, and as fB to FD.

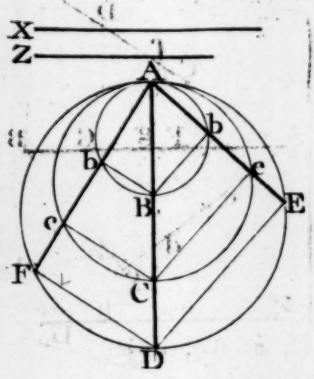
The laſt method is eaſily deduced from Prob. 32. and the Demonſtration of it, from Prop. 2. 6. El.

APPL. It is almoſt needleſs to give an Application of this Problem, as it ſpeaks its uſe ſufficiently. It is extremely uſeful in the practice of Perſpective, as well as in all geometrical Drawings; as Plans, Elevations, &c.

A B is conſidered as a finite Line, already drawn, in ſome Plan, &c. and the Ratio or Proportion given, is certain known meaſures, or diviſions, to be repreſented on AB, from either a greater or leſs Scale of Proportion.

3. There is another method, by which a Line may be divided from a greater Meaſure given; which, not ſo much for its utility as the ſingular elegance of it, I ſhall give, as follows.

Let AD be divided in the given Ratio, at B and C; and let X or Z be a Line given, to be divided, in equal Ratio.



Deſcribe three Circles, on the three Diameters AB, AC, and AD.

Take the Line X or Z in your Compaſſes; and, ſeting one Point in A, cut the greateſt Circumference at E or F, with the other Point, and join AE or AF; which, will be divided in the ſame Ratio as AD, in b & c.

Draw Bb and Cc.

DEM. The Angles A b B, A c C, alſo A E D or A F D, are Right. - - - - - - P. 12. 3. El. wherefore, A b B, A c C, &c. are ſimilar Triangles, the Angle at A being common to them all. - - C. 3. 2. 6. Therefore, A E or AF is divided in the ſame Ratio as AD, the Lines B b, C c, and ED, or FD, being parallel.—2. 6.

4. A Right Line may be accurately divided into any number of equal Parts, by the following Method.

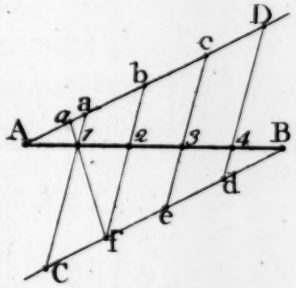
Let AB be the given Line, to be divided into five equal Parts.

From

From either extreme, as A, draw AD, making any Angle, BAD, at pleaſure; and, at the other extreme, B, make the Angle ABC equal BAD.

On AD and BC take, at pleaſure, four equal Diviſions, from A and B, at a, b, c, D and, d, e, f, C.

Join, a C, b f, &c. as in the Figure, which will divide AB into five equal Parts, as required; at 1, 2, 3, and 4. Q. E. F.



This is very evident from the ſecond method; for, the Diviſions on A D and B C being equal, they are conſequently equal on AB; the Lines A C, b f, &c. being parallel (C. 15. 1.) for, AD is parallel to B C; by P. 4. 1.

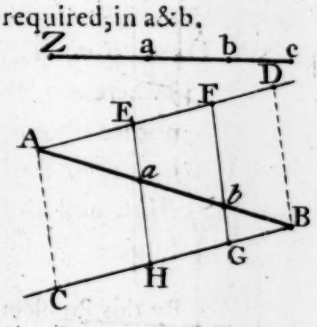
N. B. If the Diviſions on A D and B C were either greater or leſs, AB would be divided the ſame; which is obvious by taking A *a* 3 fourths of A a, and joining a f; for, B f is alſo 3 fourths of B C.

I have never ſeen this method uſed for dividing a Right Line in any given Ratio; which may be applied with ſucceſs.

Let AB be a Line given to be divided; and Z, a Meaſure known, i.e. a Right Line divided in the Ratio required, in a & b.

Make AD and BC in equal Angles with A B, as before.

Tranſer the meaſures Z a, a b, and b c, to AD and B C, in the order required; viz. make AE equal Z a, and E F equal a b. alſo, make B G equal b c, and G H equal a b, &c. and join the Points E H, F G; which will divide AB, in the Ratio of Z, in the Points *a* & *b*.



Note. The two extreme Diviſions FD and C H are of no uſe in the operation; but AC and DB, being joined, will be parallel to EH & FG; and exhibits the former Method, applied on both Sides.

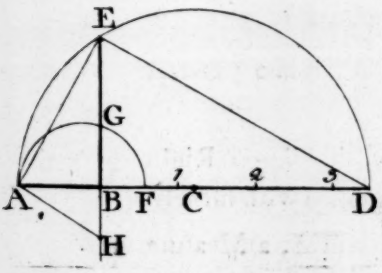
The only difference is, that, in this Method, there is no regard had to the paralleliſm of the Lines E H and F G, but only to join the Points, and they are neceſſarily parallel.

PROBLEM XXXVII.

To find the Side of a Square, or any other Right-lined Figure (ſimilar to a given one) which ſhall be to the given Figure in any proportion required.

Let AB be the given Line, or Side of the given Figure.

It is required to find the length of a Line, for a correſponding Side; on which, if a ſimilar Figure be conſtructed, its Area ſhall be to the given one in the Ratio of $3\frac{1}{2}$ to 1.



Produce the given Line, AB, indef.
Make BD equal to $3\frac{1}{2}$ times AB.
Biſect AD; and, on the point of biſection, C, deſcribe a Semicircle, or the Ark AE only.
At the Point B, draw BE perpendicular to AB, cuting the Ark at E;
BE is the Line ſought. Q. E. F.

DEM. For, BE is a mean Proportional between AB & BD.
Wherefore, the ſquare of AB, to the ſquare of BE, is duplicate of AB to BE. - - - - P. 10. & 12. 6.
i. e. their Ratio or Proportion, to each other, is as AB to BD; and all ſimilar Figures are in the ſame Ratio, as the Squares of their correſponding Sides. - - C. 2. 13. 6.

By this Problem, any right-lined Figure, whatever, may be increaſed or diminiſhed in any Proportion. e. g.

If you would decreaſe it a fifth, a fourth, a third, or a half, &c. make BF to AB in that Ratio; biſect AF, and deſcribe the Semicircle AGF, cuting the Perpendicular BE (at the Point B) in G; and BG is the Line ſought.

Hence the Ratio between any two Figures, may be known.

For,

For, having, by Prob. 24. and 25. reduced the given Figures to Squares; i. e. having found the ſide of a Square, equal to each Figure, reſpectively, make AB and BE reſpectively equal to them, and forming a Right Angle, ABE.

Produce AB or EB, indefinite; join AE, and make AED, or EAH, a Right Angle; i. e. draw ED, or AH, perpendicular to AE, cuting AB, or EB, produced, in D or H.

Then, as AB is to BD, or, as HB to BE, ſo is one Figure to the other;—by Cor. 1. P. 13. 6. El.

For, the Squares of AB and BE are, reſpectively, equal to thoſe Figures.

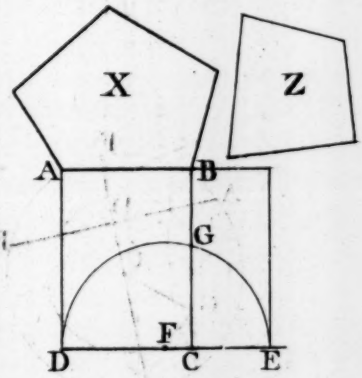
N. B. If the Figures, whoſe Ratio is required, are ſimilar, make AB and BE, reſpectively, equal to any two correſponding Sides, and proceed as above.

PROBLEM XXXVIII. 25. VI. Euclid.

To conſtruct a Poligon ſimilar to a given one, and equal to any given Right-lined Figure.

X and Z are the two Figures given.

It is required to make a Pentagon ſimilar to X, and equal to the Trapezium Z.



On any Side of the Pentagon, as AB, make a Rectangle ABCD equal to the Pentagon; by Prob. 24, 20, and 23.

Then, on the Side BC, of the Rectangle BD, make another Rectangle, BE, equal to the Trapezium; by the ſame Problems.

Biſect DE, in F; and, on that Center, deſcribe a Semicircle, with the Radius FD, cuting BC, at G.

On CG, if a Pentagon be conſtructed, ſimilar to X, making CG the correſponding Side to AB, it will be equal to the given Trapezium Z. Q. E. F.

DEM. For

DEM. For, CG is a mean Proportional between DC & CE. Wherefore, ſince the Ratio of any two ſimilar Figures, is duplicate of their correſponding Sides (13. 6.) and the Ratio of DB to BE is as their Baſes, DC to CE; - 1. 6. Conſ. the Ratio of the two Pentagons, will be as DC to CE; for, they are reſpectively equal to the Rectangles DB & BE. Therefore, a Pentagon conſtructed on CG, ſimilar to X, will be equal to the Trapezium Z. Q. E. D.

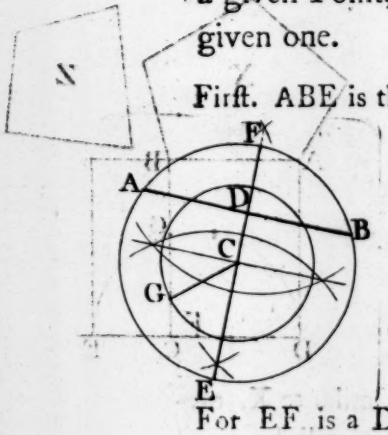
By the 24th any Figure is readily reduced to a Triangle, by the 20th a Rectangle is found equal to a Triangle; and by the 23d, another Rectangle may be found equal to that, having one Side equal to AB, or any given Line.

For, it will produce AD, for the other Side of the Rectangle AC, which is a fourth Proportional to AB and the two Sides of that Rectangle, found by the 20th, equal to the Pentagon; as by the laſt method of Prob. 32.

Conſ. the Rectangle BD is equal to the Pentagon X, and BE to Z.

PROBLEM XXXIX. 1. III. Euclid.

To find the Center of a given Circle; and through a given Point, to deſcribe a Circle parallel to the given one.



Firſt. ABE is the Circle given; and G the given Point.

Draw a Chord Line, AB, at pleaſure. Biſect AB (Pr. 8.) and through D, the point of biſection, draw EF perpendicular to AB (6.) cuting the Circumference in E & F. Biſect EF, in C, which is the Center of the Circle ABE. Q. E. F.

For EF is a Diameter, and conſequently paſſes through the Center of the Circle. - - - - - P. 1. 3.

2nd. Join

2nd. Join CG. With the Radius CG, and on the Center C, deſcribe a Circle GH, and you have done. Q. E. F.

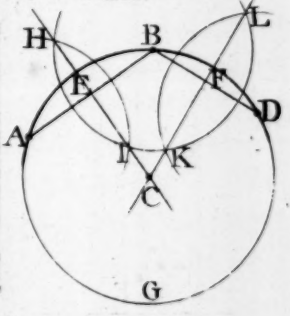
SCHOL. *All parallel Circles, in the ſame Plane, have the ſame Center, and are called concentric; but, if they are in parallel Planes, as on a Cylinder, Cone or Sphere; a Right Line perpendicular to thoſe Planes, if it paſſes through the Center of one Circle, it will paſs through the Centers of them all; which line is the Axis of the Cylinder, Cone or Sphere.*

PROBLEM XL. 25. III. Euclid.

To perfect or compleat a Circle, from a given Ark or Segment of that Circle.

A B D is the Ark given.

Draw, at pleaſure, two Chord Lines; AB and BD. Biſect the two Chords, at E and F; from which, draw the Perpendiculars EC and FC, interſecting at C. - - Pr. 8. & 6. Then will C be the Center of the Circle; on which, with the Radius CA, C B, or CD, the Circle ABDG may be compleated



This Problem is demonſtrated in the laſt.

N. B. By this method, may be found the Center of a perfect Circle, as readily as in the foregoing; nor is it neceſſary to draw the Chords; only, aſſuming, at pleaſure, three points, A, B, & D, and proceed after the manner following.

With any Radius, at diſcretion, on B, as a Center, deſcribe the Ark HIKL; and, with the ſame Radius, on A and D, deſcribe two Arks, H I and K L, cuting the former in H, I, K and L; through which Points, draw Right Lines, HI and L K, interſecting at C, the Center ſought.

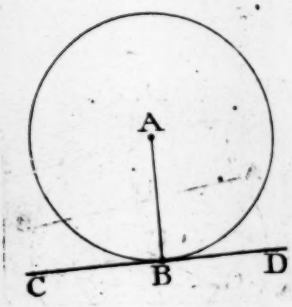
Cor.

Cor. Hence, through three given Points (not lying in a Right Line) as A, B, and D, the Circumference of a Circle may be deſcribed. whoſe Center (being found, as above) is C.

PROBLEM XLI.

To draw a Tangent to a Circle, through a given Point in the Circumference. And, to find the Point of Contact of a Tangent to a Circle.

Firſt; B is the Point given; through which a Tangent is required to be drawn.



Having found A, the center of the Circle (by the foregoing) join the point B, and the center of the Circle by a Right Line, AB.

At the Point B, make a Right Angle, ABC, and produce CB, towards D.

The Right Line CD will touch the Circle, in the Point B. - - P. 8. 3. El.

2nd. CD is the Tangent given; to find the Point of Contact.

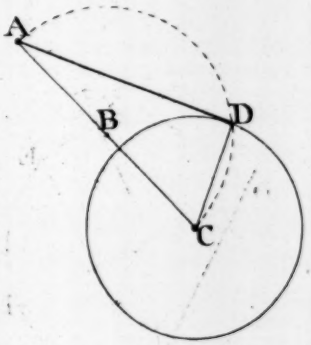
Draw a Perpendicular AB to the Tangent, from the center of the Circle; cuting CD in B; - - - - - Pr. 7.

B is the Point of Contact, in which the Tangent, AD, touches the Circle. - - - - - C. 3. 8. 3.

PRO-

PROBLEM XLII. 17. III. Euclid.

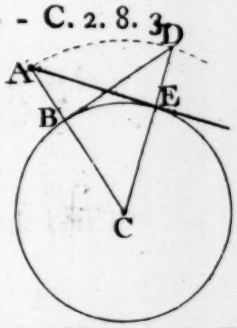
To draw a Tangent to a given Circle, from a Point given without the Circle; i. e. to determine the Point, in which a Right Line drawn from the given Point ſhall touch the Circle.



From the given Point, A, draw AC; to the Center of the Circle.
Biſect AC; and, on the point of biſection, B, with the Radius AB, deſcribe a Semicircle; cuting the circumference of the given Circle in D, the Point ſought. Q.E.F.

DEM. Having drawn AD and CD, the Right Line AD will touch the Circle at D.
For, the Angle ADC is a Right one. - - P. 12. 3.
Wherefore, AD touches the Circle in D. - - C. 2. 8. 3.

Otherwiſe.



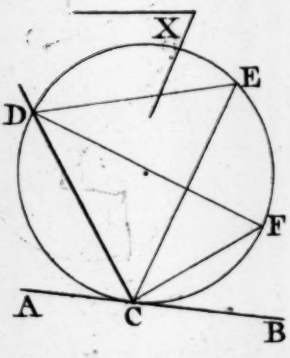
Join the Point A, and the Center C, as before, cuting the Circumference in B.
With the Radius CA deſcribe the Ark AD.
Draw the Perpend. BD, cuting AD at D.
Laſtly, draw CD, cuting the given Circle at E, the Point ſought. Draw AE.
The Right Line AE will touch the Circle, at E.

DEM. The Triangles AEC, BDC are congruous.
For, the ſides AC, CE are equal to DC and CB, reſpectively; and the Angle C is common to both; therefore, AE is equal to BD; the Angle at A equal D, and the Angle AEC equal DBC. - - - - - P. 8. 1. El.
But CBD is a Right Angle (Con.) wh. AEC is a R. Angle.
Therefore, AE touches the Circle at E. - C. 2. 8. 3.

PROBLEM XLIII. 34. III. Euclid.

To cut off a Segment of a Circle, which ſhall contain an Angle equal to a given one.

DEC is the Circle given, and X the given Angle.



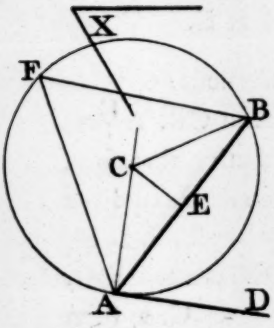
Draw a Tangent, AB, to the given Circle. At the Point of contact, C, make the Angle ACD equal to X, the Angle given; and DEFC is the Segment required. Q E F.

Dem. For, if from any Point in the Circumference, as E or F, Lines are drawn to the extremes of the Chord CD; the Angle DEC or DFC, is equal to ACD, (equal X, by Con.) - - - P. 13. 3.

PROBLEM XLIV. 33. III. Euclid.

On a given Line, to deſcribe a Segment of a Circle, which ſhall contain an Angle equal to a given one.

AB is the given Line, and X the given Angle.



Make an Angle, BAD, equal to the given Angle, X. - - - - - - Pr. 4. Draw AC perpendicular to AD, - - 10 and, on the other Extreme, B, make the angle ABC equal BAC.

Or, having biſected AB, draw EC perpendicular to AB, cuting AC in C; on which, with the Radius CA or CB, deſcribe the Ark AFB, the Segment required.

Or

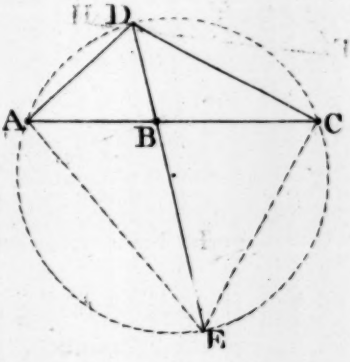
DEM. From any Point, as F, draw FA and FB.
The Ang. AFB=BAD, eq. to the given Ang. X.-13.3.

Or, without drawing AD, on the given Line AB, make an Iſoſceles Triangle, whoſe Angles at the Baſe, BAC, ABC, are each equal to the Complement of the given Angle, to a Right Angle.

The Vertex C will be the Center, and CA, or CB, the Radius of the Circle required.

PROBLEM XLV.

The Angles being given, under which three Objects, ſituate in a Right Line, are ſeen, and their Diſtances from each other known; to determine the Point from which they are ſeen.



A, B, and C are the three Objects.

Make the Angles ACD, CAD, alternately, equal to the given Angles, i. e. make ACD equal to the Angle under which AB is ſeen, and CAD equal to the other Angle given.
Produce CD and AD, interſecting at D.
Deſcribe a Circle through the two extreme Objects, A & C, and the Angle D; - - by Pr. 40.
Draw DB, and produce it, till it cuts the oppoſite Circumference, at E, the Point ſought.

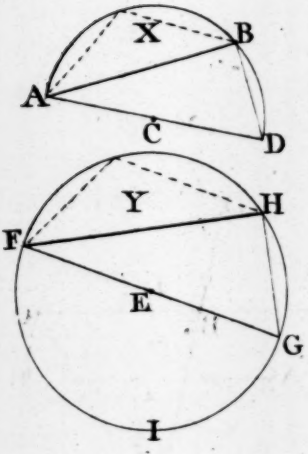
DEM. For (having drawn AE and CE) the Angle AEB=ACD, and the Angle CEB=CAD. - P. 10. 3.

PROBLEM XLVI.

A Circle being given, to cut off a Segment, ſimilar to a given Segment; and, on a given Line, to conſtruct a Segment, ſimilar to a given one.

Firſt. It is required to cut off, from the given Circle FHI, a Segment ſimilar to X.

Find the Center, C, of the given Segment, X, - Pr. 40. and compleat the Semicircle ABD.



Through the Center, E, draw FG, and make the Angle GFH equal DAB; - Pr. 4. The Segment, Y, cut off by the Chord FH, will be ſimilar to the given Segment.

DEM. For, the Angle GFH = BAD.-Con.
And (having joined BD and HG) the Angle FHG = ABD - - - - Ax. 9.
(for they are Right Angles; 12. 3.)
conſ. FGH = ADB - - C. 5. 10. 1.
wherefore, the Triangles ABD, FHG are ſimilar. - - - C. 2. 4. 6.
Therefore, as FG : AD :: FH : AB.-4. 6.
And, by taking away the Triangles ABD, FHG, from the Semicircles, there is left the Segment Y ſimilar to X.

N. B. If the Segment given had been greater than a Semicircle, the operation is the ſame; the Triangle GFH being added to the Semicircle FIG, inſtead of taking it away from FHG.

2nd. FH is the given Line, on which to conſtruct a Segment as required.

Make the Angle HFG equal to BAD; which, AB, the Chord of the given Segment, makes with the Diameter AD; found as above.
Draw HG perpendicular to FH, cuting FG in G.
Biſect FG, and on E, the Center, with the radius EF or EG, deſcribe the Ark FCH, forming the Segment required.

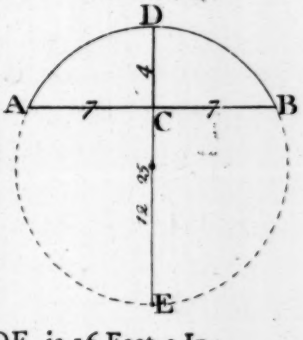
APPL. This Problem is very uſeful to Builders, to form Arches, and ſcheme Heads for Windows, Doors, &c. ſimilar to others, either greater or leſs, of any proportion required.

The following is an univerſal Rule, to find, arithmetically, the Diameter of a Circle, from any given Segment, by the meaſure of the Chord Line or Subtenſe; i. e. the Baſe of the ſegment, and the height of the Ark.

ADB is the Segment given.

Divide the Square of half the Baſe by the perpendicular height and add the Quotient to the Diviſor. See Caſe 2.14.3.

e. g. Let the Baſe AB be 14 and the Perpendicular CD, 4.
The Square of AC or CB, 7 multiplied by 7 is equal 49; which, divided by 4, gives the Quotient 12, 25, equal CE; to which add the diviſor, 4, the height of the Ark, CD, it gives 16, 25, for the Diameter.
For a Rectangle under CE and CD is equal to the Square of AC. - - - - P. 14. 3.



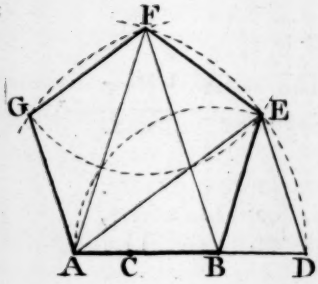
Suppoſing the Integers Feet, the Diameter, DE, is 16 Feet 3 In.

The Rule is the ſame, if the Segment be greater than a Semicircle. But, having found the Center (by Pr. 39 or 40) a Right Line drawn through the Center, cuting the Ark in two Points, is a Diameter.

PRO-

PROBLEM XLVII.

To make a regular Pentagon, on a given Line, AB.



Divide AB in extreme and mean Proportion, in the Point C. - - Pr. 35. Produce AB; and make BD equal to BC, the greater Segment; then, AD is to AB, as AB to BC, and as BC to AC.*

On A, with the Radius AD, deſcribe the Ark DEF; and, on B, deſcribe the Ark GF, interſecting at F.

On F, with the Radius AB, deſcribe the Ark EG; i. e. make FE and FG each equal AB, and join the Points A and G, B and E; alſo E, F and F, G, which compleats the Pentagon. Q. E. F.

* The Demonſtration of this conſtruction of a Pentagon may be obtained, from the 7th and 8th Prop of the 4th of Elements.

For, AED is an Iſoſceles Triangle, having its Angles at the Baſe, ED, each double the Angle EAD, at the Vertex; AFB is the ſame, which may be conſidered as inſcribed.

And, by 34. of the 6th. the Diagonal of a Pentagon, BF or AE (equal AD) has that ratio to the Side, AB, BE, &c. as the greater Segment to the leſs, of a Line divided in extreme and mean Proportion.

Or it may be conſtructed thus.

Having found the Point D, as above, and drawn the Ark DEF, on the Center A.

On B, with the Radius AB, deſcribe the Ark AE, cuting the other at E, and draw AE and DE.

Draw BF parallel, to DE; and alſo AG, indefinite.

Draw FG parallel to AE, cuting AG in G, and join FE and BE, which compleats the Figure.

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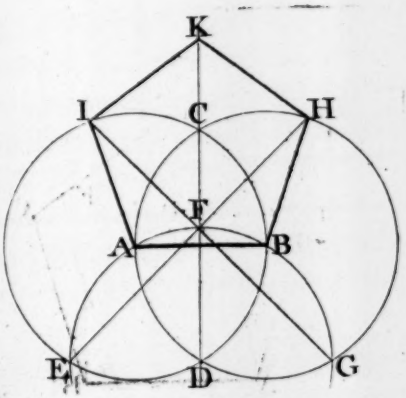
* See N. B. 2. Prop. 35. 6th of Elements.

Otherwiſe,

Otherwiſe, mechanically; AB being the given Side.



On A and B, with the Radius AB, deſcribe two Circles, cuting each other in C and D, and draw CD.

On D, with the ſame Radius, deſcribe the Ark EABG, cuting the two Circles and the Right Line CD, in E, F, and G. Draw the Right Lines EF and GF, and produce them till they cut the Circumferences, in H and I.

Then, on H and I, with the Radius AB, deſcribe two Arks, interſecting at K, and join the Points AI, IK, KH, and HB, which compleats the Pentagon, AIKHB. Q. E. F.

Of this conſtruction of a Pentagon, there is no Demonſtration has yet been given.

I ſhall, here, give a general Method for conſtructing every kind of regular Poligon, from a Pentagon to a Duodecagon, by means of the following Table; in which, the Angle of the Poligon is determined by the proportion it has to a Right Angle, and the difference there is between them.

A Right Angle is to the Angle of a Poligon as follows,

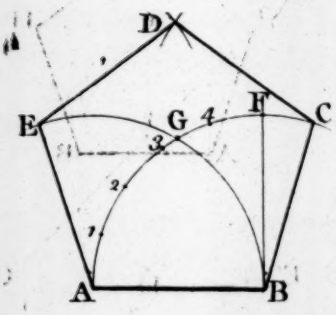
	Ratio	Diff.		Ratio	Diff.
Pentagon	as 5 to 6	— 1	Nonagon	as 9 to 14	- 5
Hexagon	as 3 to 4	— 1	Decagon	as 5 to 8	- 3
Heptagon	as 7 to 10	— 3	Undecagon	as 11 to 18	- 7
Octagon	as 2 to 3	— 1	Duodecagon	as 3 to 5	- 2

Note; the Hexagon, Octagon, &c. having an equal number of Sides, are reduced to the loweſt Denomination; otherwiſe, the Right Angle is always ſuppoſed to be divided into the ſame number of Parts as the Poligon has Sides; which will then be, for a Hexagon, as 6 to 8, difference 2; in an Octagon, as 8 to 12, difference 4; and ſo of the others.

To

To conſtruct a Pentagon, by the Table, on the Line AB.

Make a Right Angle, ABF, on the extreme Point B.



By the Table, the Angle of a Pentagon is to a Right Angle in the ratio of 6 to 5, difference 1.

With the Radius AB (or any other) on B deſcribe an Ark from A to C.

Divide the Ark (of the Right Angle) AGF into five equal Parts, as in the Figure; the Angle of the Pentagon is ſix of thoſe Parts.

Add the difference, 1, from F to C, and draw BC.

The Angle ABC is the Angle of a Pentagon, containing 6 fifths of a Right Angle, on the Ark AGC.

On A, with the ſame Radius, AB, deſcribe the Ark BGE, cuting the other at G; make GE equal GC, and draw AE.

With the ſame Radius, on E and C, deſcribe two Arks, interſecting at D, and join CD and DE; which compleats the Pentagon.

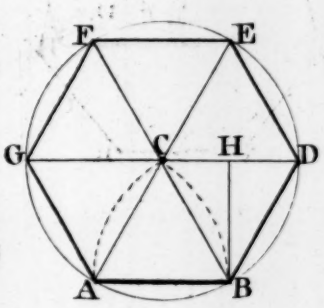
Dem. The Sides AB, BC, &c. are equal, by Conſtruction. And the Angles are alſo equal, being ſubtended by equal Arks, AGC, BGE, &c. of equal Circles. - C. 2. 9. 3. Therefore, ABCDE is a regular Pentagon.

PRO-

PROBLEM XLVIII.

To make a Hexagon, on a given Line, AB.

In the conſtruction of a Hexagon you need not regard the Table, nor proceed after that method; ſeeing that, the Side of a Hexagon is always equal to the Radius of a circumſcribing Circle. - - - - - - 11. 4. El.



Therefore, with the Radius AB, deſcribe two Arks, on A and B, interſecting at C, and compleat the equilateral Triangle ACB. On the Center C, with the ſame Radius, deſcribe a Circle.

Produce AC to E, and BC to F; and, through the Center C, draw GD parallel to AB, and join the Points A, G, F, E, D, and B.

Or, having found the Center C, and deſcribed a Circle, as above.

Apply the given Side, AB (equal to the Radius, AC) ſix times round the Circumference, from B to D, E, F, G, and join the Points as before.

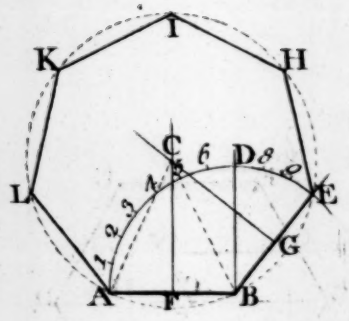
AGFEDB is a regular Hexagon. Q. E. F.

The circumference of a Circle contains the Radius exactly ſix times inſcribed (Th. P. Ang. Art. 5.) conſequently, it is equal to the Side of a Hexagon inſcribed. - P. 11. 4.

This needs no other Demonſtration; for, the Sides are all equal by Conſtruction; and the Triangles ACB, BCD, DCE, &c. are Equilateral, whoſe Angles are alſo equal (C. 1. 9. 1.) Each Angle of the Hexagon, contains two Angles of a Triangle, ABD, equal ABC added to CBD, &c. which are therefore equal; and conſequently have that proportion to a Right Angle as 4 to 3; as the Perpendicular BH, biſecting the Angle CBD, indicates.

PROBLEM XLIX.

To make a Heptagon, on a given Line, AB.



Make a Right Angle ABD; and divide the Ark of it into ſeven equal parts, as many as the Figure has Sides.

Obſerve, by the Table, that the angle of a Heptagon is ten of thoſe Parts, the difference is three.

Set off three Parts from the Perpendicular BD to E, and draw BE. Make BE equal AB. Biſect the two Sides AB and BE; - Pr. 8. and draw the Perpendiculars, FC and GC, interſecting at C. On C, the Center, and Radius CA, or CB, deſcribe a Circle; which will contain the given Line, AB, ſeven times in its Circumfnrence, at A, B, E, H, I, K, L; which Points, joined by Right Lines, will compleat the Heptagon required. Q. E. F.

DEM. It is equilateral by Conſtruction; and it is alſo equiangular, becauſe inſcribed in a Circle; for, equal Segments contain equal Angles. - - - P. 10. 3.

N. B. It is plain that the Angle ABE, which is an Angle of the Heptagon, is equal to the Right Angle ABD added to the Angle DBE; the Right Angle containing ſeven parts, the Acute Angle, DBE, three; and conſequently, the Angle ABE contains ten; therefore, the Ratio is as 10 to 7; as by the Table.

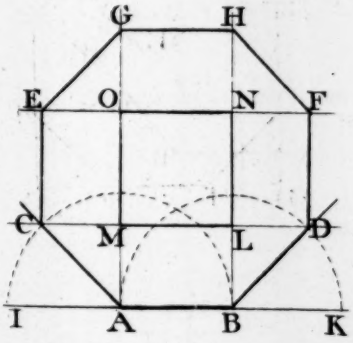
After the ſame manner a Circle may be found, which ſhall contain a given Line any number of times, to twelve, applied to the Circumference, by the Table, given above.

PRO-

PROBLEM L.

To deſcribe an Octagon, on a given Line, mechanically, without the Table.

On the extremes of the given Line, AB, draw the Perpendiculars AG and BH, indefinite.
Produce AB, both ways, to I and K.
Biſect the external Angles, IAG & HBK, by the lines AC and BD; which, make equal to AB, and draw CD; which, will be parallel to AB.
Make LN equal LM; and, through N, draw EF parallel to CD.
Draw CE & DF, parallel to AG & BH.
Make NH equal NF, and draw GH parallel to EF; and join EG and FH; which compleats the Octagon. Q. E. F.

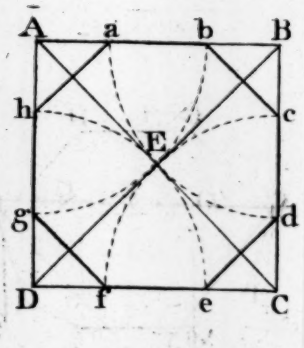


N. B. It is obvious, in this Figure, that its Angles, as ABD, has the proportion to a Right Angle, ABL, of 3 to 2, as in the Table; for, the Angle LBD is half the Right Angle LBK; and LBD added to ABL, equal ABD, is the Angle of the Octagon.

PRO-

PROBLEM LI.

To find the Side of an Octagon, in a given Square, ABCD.



Draw the two Diagonals AC and BD, which gives the Center E. - P. 16. 1. El. With the Radius of half the Diagonal, AE, on every Angle of the Square, deſcribe an Ark or Quadrant, bEg, aEd, &c. i. e. make Ab, Ag, Ba, Bd, &c. equal half the Diagonal; and, joining the Points a h, bc, &c. you will have a regular Octagon a b c d e f g h. Q. E. F.

APPL. The uſes of theſe Problems, to various Mechanics, are very obvious. By the 50th we learn how to form an Octagon Building; as a Temple, Library, &c. an Alcove, or Bow Window, &c. (which are frequently half a regular Octagon or Hexagon) of a given meaſure, for a Side of the Building, &c.

By this Problem, we find the meaſure of a Side, when the width or Diameter is firſt determined.

To enumerate all the uſes of Poligons, would be impertinent and foreign to the purpoſe; my deſign being to ſhew how to conſtruct them, in the eaſieſt and readieſt manner, the application of them will readily occur, as occaſions require.

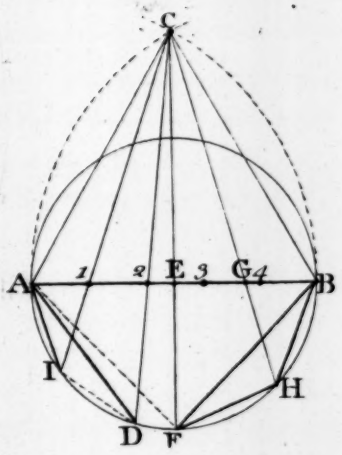
Theſe are the moſt neceſſary Poligons for mechanical uſes; and ſince, by the Table, and Rules already given, any Poligon, to twelve Sides, may be readily conſtructed, I ſhall defer treating more fully on them to the fourth Book; which teaches how to inſcribe and circumſcribe all kinds of regular Figures.

Although it is entirely Problematical, yet I think it beſt to follow the order of Euclid, in that; ſeeing, it cannot be properly treated on without the Elements of the firſt three Books. The practical Part is, nevertheleſs, eaſy and intelligible to any Capacity.

PRO-

PROBLEM LII.

To find the Side of any regular Poligon in a given Circle.



Draw a Diameter, AB, on which, conſtruct an equilateral Triangle, ACB; or draw the two Arks only, interſecting at C. (This preparation is the ſame for any Poligon whatever.)
Then, divide the Diameter into as many equal Parts as the Poligon, required, has Sides; (Pr. 36.) and, through the ſecond diviſion, from either extreme, draw a Right Line, from C to the oppoſite Side, of the concave Circumference. e. g.

If a Pentagon be required, the Diameter (AB) muſt be divided into five equal Parts; through the ſecond Diviſion, from A or B, draw CD; then, AD is the ſide of a Pentagon; i. e. a fifth part of the whole Circumference, or two fifths of the Semicircumference AFB.

The Side of a Hexagon is equal to the Radius; and the Side of an equilateral Triangle is the Diagonal of two ſides of a Hexagon; yet the ſame Rule holds equally true in all.

If a Right Line, be drawn, from C, through the Center, E, to F, it divides the Circle equally into four, and AF or FB, is the Side of a Square.

For, if the Diameter be divided into four equal Parts, the Center being equally diſtant from each extreme of the Diameter, AE is, conſequently, two of thoſe Parts.

One

One of thoſe Parts, GB, equal half the Radius, EB, is two eighths of the Diameter; wherefore, if CG be drawn and produced, it will cut the concave Circumference in the Point H; and BH, or FH, being joined, is the Side of an Octagon; for it will biſect the Ark, FHB, of the Side of a Square.

Again. A1, being a fifth part of the Diameter, is equal to two tenths; wherefore, if C1 be drawn, to I, it will biſect the Ark AID; and AI or ID, being joined, is the ſide of a Decagon, inſcribed; for, it is equal to two tenths of the concave Circumference AFB.

Thus may the Side of any Poligon whatever, contained in a Circle, be obtained; by obſerving the Rules given above. And it is truly worthy of notice; that, any Right Line, drawn from C, cuting the Diameter and the concave Circumference, will cut them both in the ſame Proportion; or, in whatever Ratio one of them is divided, a Right Line being drawn, from C, through the point of diviſion, will alſo cut the other in the ſame Ratio.

Of this Conſtruction, or equal diviſion of the Diameter and the Circumference, no Demonſtration can be given, having conſulted ſeveral able Geometricians concerning it; who ſay, that it is only an approximation and not mathematically true. Yet, I muſt own, that I do believe it to be perfectly true, or it could never anſwer ſo very accurately, as it does, in all Diviſions, whatever.

OF

OF THE

ELLIPSIS.

HAVING now, gone through all the moſt uſeful and valuable Problems in Geometry; I ſhall, next, before I proceed to the Elements, ſhew how to deſcribe that uſeful and elegant Figure called an Ellipſis; which is ſo very neceſſary to Mechanics, and particularly to Architects and Builders, in general; inſomuch, that I ſhould reckon a compleat Syſtem of practical Geometry deficient without it.

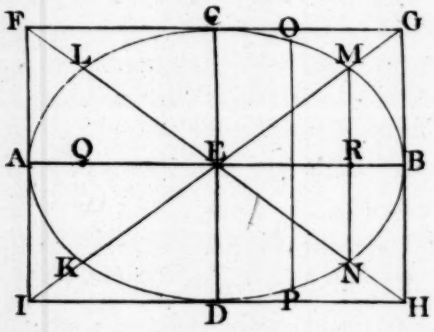
The Ellipſis, is a Figure which is not admitted into Plane Geometry; as the nature of its Curve and the properties peculiar to it, are of no uſe in the Elements of Euclid. Nevertheleſs, as it is a very uſeful Figure, I ſhall define all its parts, particularly, and ſhew how, by various ways, it may be deſcribed of any Proportion required; a thing much wanted and but little known, to many who have a conſtant occaſion for it; how many lame and imperfect Ovals may be ſeen amongſt the Works of our moſt eminent Artiſts, at an Exhibition, is but too obvious to a judicious Eye.

I ſhall alſo explain ſome of its peculiar Properties; ſhewing the affinity between the Circle and Ellipſis (as between a Square and Rhombus) the Ellipſis being conſidered as a Circle, preſſed gently, or drawn out, at the two extremes of any Diameter.

DEFI-

DEFINITIONS.

DEF. I. An ELLIPSIS, or OVAL, is a Plane Figure bounded by a regular curved Line, falling into itſelf, which, is not circular in any part, being deſcribed on two Centers.



DEF. II. PERIPHERY, or CIRCUMFERENCE, is the curved Line which bounds the Ellipſis; ACBD.

DEF. III. CENTER, of an ELLIPSIS, is the Point E, where any two Diameters interſect, and, conſequently, biſect each other.

DEF. IV. DIAMETER, of an ELLIPSIS, is any right Line, as AB, MK, &c. paſſing through its Center, E, and terminated by the Periphery.

For, every ſuch Line divides the Ellipſis equally in two; and is alſo biſected in the Center, E. Therefore, any two Diameters biſect each other; as in a Circle.

DEF. V. TRANSVERSE DIAMETER is the longeſt which can be drawn in an Ellipſis. As AB.

DEF. VI. CONJUGATE DIAMETER. This Term is generally confined to the ſhorteſt Diameter, CD; but, that is conjugate only in reſpect of the Tranſverſe AB; which is alſo conjugate to CD.

If Tangents to the Ellipſis be drawn, through the extremes, A and B, of the Tranſverſe, and CD its conjugate Diameter, cuting each other in F, G, H and I, they will

will form a Rectangle; and if the Diagonals of the Rectangle, FH and IG, be drawn, they will paſs through the Center of the Ellipſis; the parts, KM and LN, which are terminated by the Curve, are Diameters of the Ellipſis; each of which, is Conjugate, in reſpect of the other.

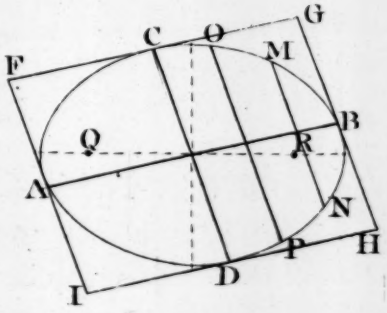
Theſe are the only two Conjugate Diameters which are equal.

The Tranſverſe Diameter, and its Conjugate, are the only two which are perpendicular to each other, or cut at right Angles.

The Tranſverſe Diameter, AB, biſects the acute Angles FEI, GEH; and the obtuſe Angles, FEG, IEH, are biſected by the Conjugate, CD.

Every Diameter, drawn within the acnte Angles, is greater than its Conjugate, which falls within the obtuſe Angles.

DEF. VII. ORDINATES are Right Lines drawn parallel to the Conjugate of any Diameter, as MN, OP, &c. and they are biſected by the Diameter, AB. The whole Lines MN, OP, are, therefore, double Ordinates to the Diameter AB.



DEF. VIII. A TANGENT is a Right Line touching the Periphery, at the extreme of a Diameter, parallel to its Ordinates. As FI, FG, &c.

DEF. IX. AXES, of an Ellipſis, are the Tranſverſe and its Conjugate Diameter; AB and CD, Fig. 1ſt.

DEF. X. FOCI, of an Ellipſis, are the Points Q and R, on which it is deſcribed; as a Circle on its Center.

DEF. XI. LATUS RECTUM, or RIGHT PARAMETER, is that Ordinate to the Tranſverſe Axe which paſſes through either Focus; as MN; Fig. 1ſt.

The Latus Rectum, and every Parameter, is a third Proportional to the two conjugate Diameters, to which it is the Parameter.

Note. The Parameter to the Diameters KM and LN, Fig. 18. coincides with them; for, being equal, there cannot be a third Proportional to them.

DEF. XII. ABSCISSA. If the Tranſverſe Diameter be cut, in any Point at pleaſure, the two Segments, made by that Section, are called Abſciſſas.

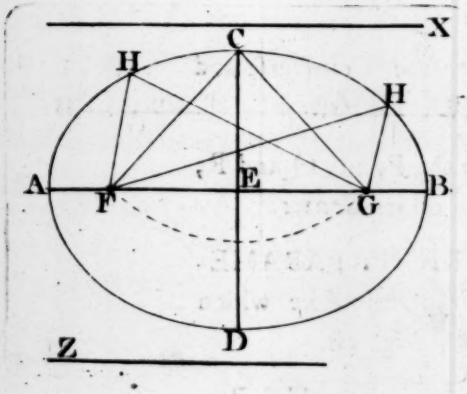
Note. An Ellipſis may be generated, by the Shadow of a Circle, or Ring, projected on a Plane, not parallel to the Ring; provided, the luminous Point is not in the Plane of the Circle.

Or it is an oblique Section, made by a Plane, of a Cylinder or Cone. (See Def. 15 and 18, 7th. El.)

PROBLEM I.

The Tranſverſe and its Conjugate Diameter being given, how to determine the Foci, on which an Ellipſis may be deſcribed, of the Proportion required.

Let X and Z be the meaſures given.



Draw, at right Angles, two Right Lines, AB and CD, equal, reſpectively, to X and Z, and biſecting each other, in the Point E.

On either extreme, C or D, of the Conjugate Diameter, deſcribe an Ark, with the radius AE, or EB, half the Tranſverſe, cuting it in F and G; which are the Foci, or Centers, on which the Ellipſis may be deſcribed.

Then, proceed in the manner following.

Take

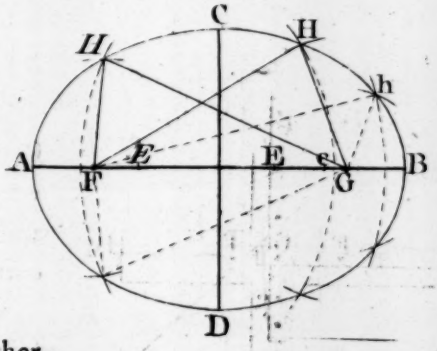
Take a fine, ſmooth Cord or ſtring, and, having fixed two Pins in F and G, carry the ſtring round both Pins, and draw it, on both Sides, to the Point C, or D; where, fix a Pencil, and revolve it around, on the two Pins, keeping the String at full ſtretch; the point C will deſcribe a Curve, which will paſs through the four Points A, C, B and D.

From which conſtruction it is evident, that any two Right Lines, FH and GH, drawn from the Foci to any Point in the Circumference, are equal to FC and CG; that is, to the Tranſverſe Diameter, AB.

For, FG being common to both the Triangles FCG and FHG, the remaining Sides, added together, are equal;
i. e. FC added to CG is equal to FH added to HG.

As it is difficult to keep the Point or Pencil true, in a ſtring, this Method is not ſo eligible, for ſmall Ovals, or for any, which require the Curve to be exact; but, from what has been advanced, (being well conſidered and underſtood) it will be found practicable to deſcribe ſmall Ovals with tolerable exactneſs.

Having determined the two Foci, F and G, in the given Tranſverſe Diameter, AB, whoſe Conjugate is CD (as above) as many Points, H, may be determined, in the Periphery, as are neceſſary; through which a Curve may be deſcribed, by a ſteady Hand, which will be a true Ellipſis; after the following manner.



With any Radius, at pleaſure, ſetting one Point of the Compaſſes in either Focus, as at F; with the other Point, make an Ark, at H. Then, with the ſame Radius, FH, ſeting one Point in either extreme of the tranſverſe Diameter, as at A, cut the Tranſverſe, at E with the other. Take the remaining Segment, EB, as Radius, and, on the Center, G, deſcribe another Ark, cuting the former in H; which will be a Point in the Periphery.

Thus, as many Points, H or h, may be obtained as you pleaſe; which are all in the Periphery.

That the Point H is in the Periphery is manifeſt. For, ſeeing that the Foci are determined from rhe extremes, C and D, of the Conjugate Axe, making CF and CG each equal to AE, half the Tranſverſe; (by Prob. 1.) conſequently, FH and GH being made alſo equal to AB, the Point H is in the Periphery.

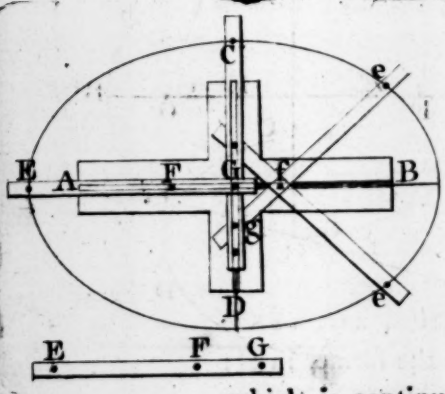
2nd. To deſcribe an Ellipſis, by means of an Inſtrument called a Trammel.

The Trammel is made of two pieces of wood, fixed together at right Angles, in the form of a Croſs; or, for ſmall Ovals, it may be of Metal, with two ſtreight Grooves, truly perpendicular to each other, as AB and CD.

Then, having provided a ſtreight Ruler, of wood or metal, fix a Point or Pencil at one end, as at E; and, let two ſhort, round Pins (equal, in Diameter, to the width of the Grooves in the Trammel, which will ſlide in them, freely) be fixed at F and G, in the manner, or order, as follows.

Make EG equal to half the Tranſverſe, and EF half the Conjugate Diameter.

The Trammel and the Ruler being thus prepared, let the Trammel be placed, as in the Figure; exactly on the Tranſverſe and Conjugate Diameters.



Then, apply the Ruler, with the Pins in the Groove, along the Tranſverſe Diameter, G being in the Center, and the point E in the Periphery, at the extremity of the Tranſverſe.

When the point E is moved towards C, the Pin at G, in the Center, falls into the Groove towards D, whilſt the Pin at F, moves towards the Center; which is continued till it falls into the Center, at G, and the point E being arrived at C, has deſcribed, by its motion, the

the Curve AC, a fourth part of the Periphery; which has all the variety of the whole; for every fourth part is the ſame; from C to B, and from A to D, it is inverted.

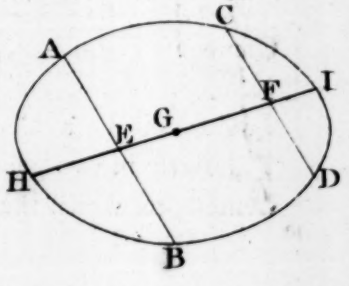
The Ruler being now in the poſition of the Conjugate Diameter; the Pin, at F, in the Center, and that at G fallen down to g; the motion is continued to B; the Pin at F, being now ſuppoſed to be in the Center, moves towards B, and the Pin at g goes back again to the Center; when the Point has deſcribed the Curve CB, another 4th part, the converſe of the former; thus continuing the motion until the Curve is compleated, and the point E arrives again at A; the Pin, at F, ſtill moving, to and again, in the Tranſverſe, from F to f and back again to F; and the Pin at G, to and again, in the Conjugate Groove; by which means a true Ellipſis is formed; which, it is evident, has no part of the Curve of a Circle in its compoſition, ſeeing it is deſcribed on two Centers.

The nearer the Foci are together, that is, the leſs the difference is between the Tranſverſe and Conjugate Diameters, the nearer it approaches to a Circle, and at laſt ends in a Circle, when the Centers unite.

PROBLEM II.

To find the Center of an Ellipſis.

Draw, at pleaſure, two parallel lines, AB and CD, in the Ellipſis. Biſect them, in E and F; through which Points draw HI, which is a Diameter, of the Ellipſis.
Biſect the Diameter, HI, in G, which is the Center of the Ellipſis.

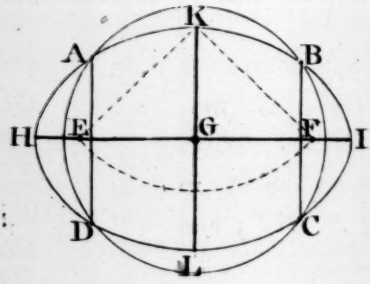


For, every Right Line drawn through the Center is a Diameter; and every Diameter is biſected in the Center.

PRO-

PROBLEM III.

To find the Axes of an Ellipſis.



Having found the Center, by the laſt, with any radius, leſs than the Tranſverſe Diameter and greater than the Conjugate, deſcribe a Circle, cuting the Ellipſis in four points, A, B, C, and D.

Draw the Ordinates AD and BC and biſect them, in E and F; through E and F draw HI, which is the Tranſverſe Axe.

Through the Center, G, draw KL, at right Angles with the Tranſverſe, or parallel to the Ordinates, AD and BC, and that is the Conjugate Axe.

PROBLEM IV.

To find the Foci of an Ellipſis.

Having found the Center, G, and drawn the Tranſverſe and Conjugate Diameters, or Axes, take GH, half the Tranſverſe, for radius, and on either extreme, K or L, of the Conjugate, as a Center, deſcribe an Ark, cuting the Tranſverſe in two points, E and F, which are the Foci, or Centers, on which the Ellipſis is deſcribed.

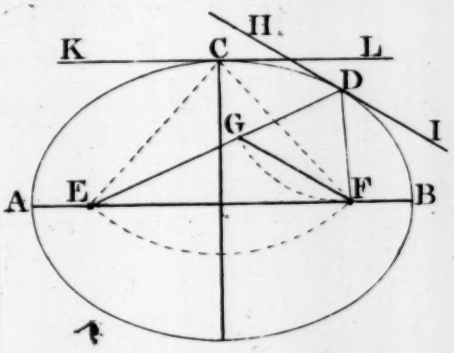
PRO-

PROBLEM V.

To draw a Tangent to an Ellipſis at any Point given; in the Periphery.

If the Tangent was required at either extreme of the tranſverſe or conjugate Axes, as B or C, it will be perpendicular to them, as in the Circle, and conſequently makes equal Angles.

Let D be the given Point, through which a Tangent is required to be drawn.



Find E and F, the Foci of the Ellipſis, by the foregoing; and draw ED and DF.
Make DG equal DF, and draw GF; to which, if HI be drawn parallel, through the given Point D, it will touch or be a Tangent to the Ellipſis in that Point.

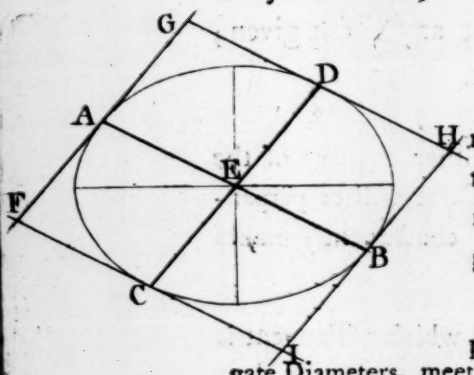
For, let KL be drawn, through C, parallel to the Tranſverſe, AB; it will touch the Periphery in C.

Draw EC & CF; they are equal, by Conſtruction - Pr. 4.
and, conſequently, the Angle ECK, is equal LCF - 4. 1. El.
for the Angles CEF, CFE are equal - - - 9. 1. El.
But, the Tri. GDF is Iſoſceles, and HI is par. to GF.
conſ. the Angle HDG or E is equal to FDI. - 4. 1. El.
Therefore, HI touches the Ellipſis, in the Point D.

PRO.

PROBLEM VI.

Any Diameter, as AB, being given, to find its Conjugate, CD.



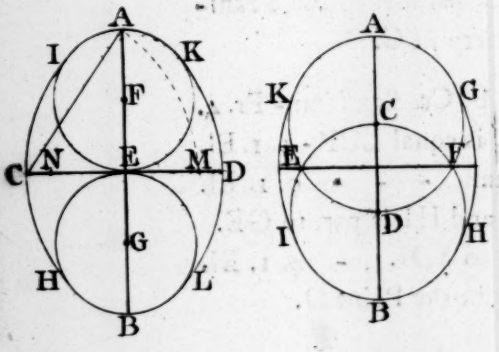
Through either extreme of the Diameter, as A, draw the Tangent FG; through the Center E, draw CD, parallel to the Tangent; which is Conjugate to the Diameter AB.

If Tangents are drawn, through the four points A, B, C and D, of any two Conjugate Diameters, meeting in F, G, H and I, the Parallelogram FGHI, is equal to a Rectangle under the two Axes.

PROBLEM VII.

To make a Repreſentation of an Ellipſis, with Compaſſes.

Let AB be the given Tranſverſe Axe, and CD the Conjugate.



Biſect AE and EB, in F and G, and on the Centers F and G, deſcribe the Circles AIEH, & EKBL, touching at E.

Draw CA; and, with that radius, on C, deſcribe the Ark AM, cutting the Conjugate Axe in M.

Make EN equal EM; on which Centers, M and N, with the radius MC and ND, deſcribe the Arks HI and KL, falling into the Circles AE and EB, at H, I, K, and L.

Or

Or thus, when the difference between the Axes is leſs.

Divide the Tranſverſe, AB, into three equal Parts, at C and D; on which, with the radius AC, deſcribe two Circles, cuting each other in E and F; on which Centers, with the radius AD, deſcribe the Arks GH and IK.

This laſt, is nearer to an Ellipſis than the other; and they may be lengthened or ſhortened at pleaſure. It is very obvious that they are not Ellipſes, but compounded of circular Curves; whereas, a true Ellipſis has no part of the curve of a Circle, in its compoſition, the Curve, being every where deſcribed on two Points, is continually varying; wherefore, no Oval, formed by Circles, can partake of the properties of an Ellipſis.

Having explained or defined all the Terms peculiar to the Ellipſis, and ſhewn how to conſtruct it of any given Dimenſions, and to fix the Points on which it is deſcribed, there remains nothing more to be done in reſpect of the utility of it, to Mechanics, &c.

To treat at large of its Properties would neither be proper nor neceſſary, in this Work, for the reaſon mentioned in the Prelude or Introduction.

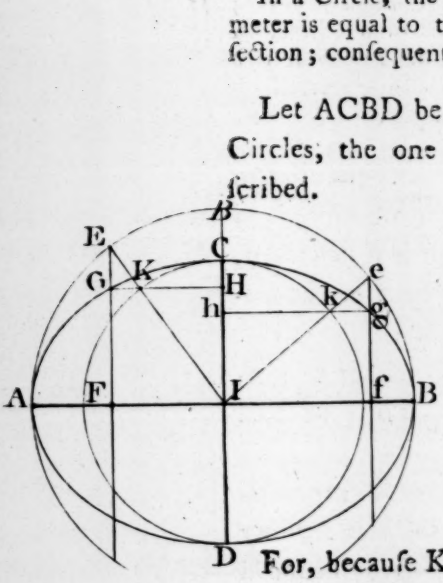
The Properties of the Ellipſis are really very extraordinary and ſurprizing; many of which have ſo near Affinity to thoſe of a Circle, that they are almoſt neceſſarily deduced from them. But, as it is requiſite to have ſome acquaintance with the Conic Sections, in order to a right and clear underſtanding of the Properties of the Ellipſis, which would not be proper to enter upon in this place; I ſhall content myſelf with giving one or two general Theorems, from which, the preceding Problems are deducible; and juſt mention two or three particular Properties, and then proceed to the Elements of Geometry, from which this may properly be called a Digreſſion, though a very uſeful one; and which, I am perſuaded, will not be unacceptable to many.

THEOREM I.

The Rectangle, under any two Abſciſſas, has the ſame Proportion to the Square of the Ordinate which divides them, as the Rectangle under any other two Abſciſſas, has to the Square of the Ordinate dividing them.

In a Circle, the Rectangle, under the two Segments of a Diameter is equal to the Square of an Ordinate, at the point of biſection; conſequently they have all the ſame Ratio (Pr. 14. 3. El.)

Let ACBD be an Ellipſis; and let there be deſcribed two Circles, the one circumſcribing it, the other, ſCFD, inſcribed.



From any Point, E, in the outer Circles Periphery, draw the Right Line EF, parallel to the Conjugate Axe, CD, cuting the Periphery, of the Ellipſis, in the Point G; from which Point, draw GH, parallel to the Tranſverſe, AB; alſo draw the Radius, EI, which will cut the inſcribed Circle in the ſame Point K.

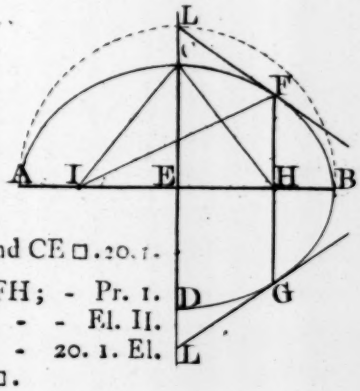
For, becauſe KG is par. to IF, FE:FG::IE:IK, - 2. 6. i. e. as IB:IC. And, EF par. to CD will be cut, in G, by the Periphery of the Ellipſis, as I*B* to IC. - 19. Em. C. Sec.

Now, GH is parallel to IB, and EF to CI - - - - Con.
wh. the Angle HKI=EIF, and HIK=IEF - - 4. 1. El.
conſequently, the Triangles, IHK and IEF, are ſimilar;
wh. as IE:EF::IK:IH. - - - - - - - - - - 4. 6.
But, AI=IE, IC=IK; and FG=IH; - - - - - - 15. 1.
wh. AI:EF::IC:FG; conſ. AI:IC::EF:FG, alternately;
and conſequently, AI□:IC□::EF□:FG□. - - - - 15. 6.
But, AF×FB is equal to EF□, - - - - - - - - 14. 3.
and, AI×IB is the ſame as AI□; for AI=IB.
Therefore, as AI×IB:IC□::AF×FB:FG□.

The ſame holds true of any Abſciſſas. whatever, and their Ordinate. For, Af×fB:fg□::AI×IB:IC□
conſ. Af×fB:fg□::AF×FB:FG□.

2. The

2. The Latus Rectum (FG) is a third Proportional to the Tranſverſe and Conjugate Axes.



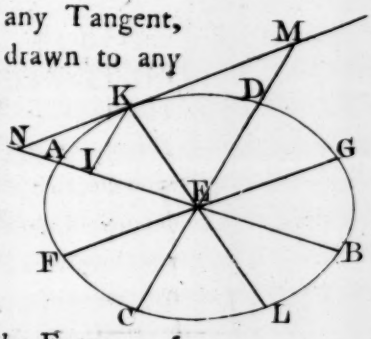
The common method of proving it, is to make FG to AB, as CE □ to AE □ ; i. e. as CD □ to AB □ (4. 2. El.) which is making FG a third Proportional to AB and CD (12 and 13. 6.) and then, proving it to be what it is by Conſtruction.

The diſtance of the Focus from the Center = the Sq. Root of the diff. between AE (eq. CH) □ and CE □. 20. 1.

Now, IF+FH=AB=2EB, i. e. IF=2EB−FH; - Pr. 1.
wh. IF □ =4EB □ +FH □ −4EB×FH. - - - - - El. II.
But, IF □ =IH □ +FH □ ; - - - - - - - - - 20. 1. El.
wh. 4EB □ +FH □ −4EB×FH=IH □ +FH □.
conſ. 4EB □ −4EB×FH=IH □ =4EH □, - - - Ax. 7.
and, 4EB □ =4EB×FH+4EH □. - - - - - - - El. II.
i. e. 4EB □ =4EB×FH+4CH □ −4CE □.
wh. 4EB □ +4CE □ =4EB×FH+4CH □.
But, CH=EB (Pr. 1.) wh. 4CE □ =4EB×FH;
i. e. 4CE □ =2EB×2FH; i. e. CD □ =AB×FG.
conſ. FG is a third Proportional to AB and CD. - - 9. 6. El.

3. If from the Center, E, of an Ellipſis, EL (on the conjugate Axe) be made equal to the Semitranſverſe, AE, the Tangents LF and LG, drawn from that Point, will paſs through the Extremes of the Latus Rectum.

4. If from the Point of Coutact, K, of any Tangent, KN, to an Ellipſis, an Ordinate, KI, be drawn to any Diameter, cuting it in I; half that Diameter, AE, is a mean Proportional between the Segment EI, made by the Ordinate, and the Diſtance of the Point N, from the Center, where the Tangent cuts the Diameter, AB, produced; i. e. EI:EA::EA:EN.

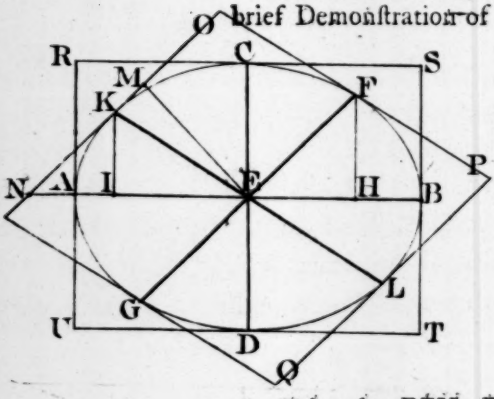


5. If a Tangent MN be drawn through the Extreme of any Diameter, KL, cuting any other two conjugate Diameters, AB and CD, in M and N; the Rectangle under KM and KN, the Segments of that Tangent, will be equal to the ſquare of EF, half the Diameter FG, conjugate to the Diameter KL.

6. If from the Extremes, F and K, of any two conjugate Diameters, Ordinates are drawn to the tranſverſe Axe, cuting it in H and I; then is the Ordinate FH to the ſemi-conjugate Axe, as the Segment EI, made by the other Ordinate, to the ſemi-tranſverſe Axe,
i. e. FH:CE::EI:EA; and IK:CE::EH:EB.

7. Every Parallelogram formed by Tangents at the Extreme of any two conjugate Diameters, is equal to a Rectangle, under the tranſverſe and conjugate Axes.

As this is a curious and extraordinary Property, I ſhall give a brief Demonſtration of it, as follows.



Let NOPQ be a Parallelogram, made by the Tangents, NO, OP, &c. at the extremes of the Diameters GF and its Conjugate KL.

Draw the Ordinates, FH and KI, to the tranſverſe Axe, cuting it in H and I; and draw EM perpendicular to NO; and conſequently,

the Triangles EFH, ENM are ſimilar. - - - - - 4. 6. El.
for, the Angles NME, EHF are Right; and MNE=FEH. 4. 1.
Wherefore, - EF:FH::EN:EM. - - - - - - by 4. 6.
Now, - - FH:CE::EI:EA (6.) and EI:EA::EA:EN (4.)
wherefore, - FH:CE::EA:EN; by equality of Ratios.
Then, ſince, EF:FH::EN:EM; and, FH:CE::EA:EN;
conſequently, EF:CE::EA:EM; by inordinate equality;
wherefore, - EF×EM=CE×EA - - - - - - by 9. 6. El.
But, - CE×EA = ¼ of the Rectangle under AB and CD.
And, - - EF×EM = ¼ of the Parallelogram NOPQ. - 18. 1.
Therefore, the Par. NOPQ is equal to the Rect. RSTU. Ax. 5.

ELEMENTS

OF

GEOMETRY.

BOOK I.

THIS firſt Book of Elements treats of particular properties of Right Lines and right lined Figures, reſpecting themſelves, and alſo their relation to, or connection with one another; of Angles, formed by the interſections of Right Lines, &c. which, being entirely unconnected with Figures, in this Work, are firſt treated on, before we begin with Figures.

Next we proceed with Triangles, the firſt, yet moſt uſeful and extenſive, of all right lined Figures. Then is ſhewn the affinity between Triangles and Parallelograms, and other four ſided Figures; all which, is neceſſary, in ſome degree, to prepare the way for the laſt grand Propoſition, the 47th of Euclid, which concludes this Book.

The Order in which theſe things are treated, according to Euclid, ſeems to me not very regular; the chain of connection is frequently broken or interrupted; beginning firſt with Figures, then Lines and Angles; again Figures and Lines, alternately. The fourth Propoſition begins the properties of Triangles, in general; the 5th introduces the Iſoſceles, a particular ſpecies of Triangles, merely (in this place) for the Demonſtration of the 7th; and that, is of no other uſe but to demonſtrate the 8th, which may be clearly demonſtrated (according to Proclus) without it. In the 24th

and

25th is again introduced the ſame general properties of Triangles, which are evidently deducible from the fourth. For, having, in the fourth, proved, that, in two Triangles, if there are found two Sides, in each, reſpectively equal, and that the Angles, contained between the equal Sides, are alſo equal, the remaining Sides are equal; can any Perſon of common ſenſe, who knows what a Plane Angle means, want Demonſtration, that if, in the Caſe of equality of the Angles, the Sides oppoſite thoſe Angles are equal; if one of the Angles be greater or leſs, the ſide oppoſite will alſo be greater or leſs? and, *vice verſa* (in the 25th) if the Side or Baſe, lying between the equal Sides, be greater or leſs, the Angle it ſubtends muſt neceſſarily be ſo too: theſe things ſeem, to me, ſuch a neceſſary conſequence, that I would not heſitate to take them for granted.

This Book is much encumbered with uſeleſs Demonſtrations, of ſeveral Propoſitions, already fully demonſtrated or need none; for, what is evident in itſelf, needs no other Demonſtration; and ſuch Propoſitions, as may eaſily and clearly be deduced from ſome preceding ones, are as well made Corollaries to them; ſuch are all converſe Propoſitions; for I cannot conceive it uſeful to perplex and embarraſs the minds of Youths, with demonſtrating what is not eſſential and abſolutely neceſſary.

I have, therefore, greatly abridged, and altered the Elements of Euclid's firſt Book; yet, I am greatly miſtaken, if I have not retained all that is eſſential in it; and which, I think, may be much eaſier and more regularly acquired. After abſtracting the 14 Problems, I have but 20 other Propoſitions in this firſt Book; three of which number (the 3d the 6th and 16th) are not in Euclid; whereas, excluſive of the Problems, Euclid has 34 Theorems, twice the number, of the remaining ſeventeen.

The

The 7th of Euclid is not only uſeleſs, in itſelf, but the Demonſtration is intricate, to a beginner; and ſince the 8th. which is made to depend on it, is clearly deducible from the 4th, I ſee no uſe for it all. In reſpect of the 4th (my 8th) I cannot think the Demonſtration of it perfectly geometrical; but, ſuch as it is, has been generally accepted; and although it has been thought ſomewhat deficient, yet, I have not ſeen even an attempt at any other kind of Demonſtration; which is ſufficient conviction that no other can be given. As it is the firſt Propoſition on the properties of Figures, which might well paſs for an Axiom, the Demonſtration, being purely mental, is ſatisfactory; though it ſuppoſes a manual, or mechanical application of one to the other; the proof ariſing, from which, would be, merely, occular.

Having received the firſt rudiments of Geometry from Pardie, I may perhaps have imbibed a prejudice, in reſpect of his manner of treating the firſt Elements. Upon the whole, I look on that Tract as very imperfect and irregular; but I have always thought it more introductory to treat, firſt, of Lines and Angles, before Figures; keeping them diſtinct and unconnected, as much as poſſible. The 13th. of Euclid I have, therefore, made the firſt; for, on the foundation of it, depends, in a great meaſure, the whole firſt Book. I am the more confirmed in it, having ſeen (ſince my Plan was formed) two or three Authors, of ſome Fame, who have purſued the ſame Plan, nearly; which at firſt (I freely own) hurt me, not a little, but, which, I am now reconciled to, and think it does me credit.

According to Euclid, the properties of parallel Lines &c. cannot be obtained without having recourſe to Triangles, and is the reaſon why the 27th, and the following, are not introduced ſooner, which are previouſly neceſſary to demonſtrate the 32nd; in which, the 16th and 17th are more fully and perfectly demonſtrated: wherefore, Euclid

was

was obliged to have recourſe to the 16th, as a Lemma, (which is much more complex and difficult to conceive than the 32nd) in demonſtrating ſeveral Propoſitions previous to them. But, for what uſe the 17th is introduced, I am at a loſs to deviſe, it being never once referred to, I believe, in the whole Elements; it might (if it be neceſſary) be a Corollary after the 32nd, and it is never uſed before. We can never be at a loſs to know, that any two Angles of a Triangle are leſs than two Right Angles, having full Demonſtration (in the 32nd) that, the three Angles of every Triangle are, together, equal to two Right ones; conſequently it is of no uſe at all.

In the 18th of this, is contained the 35, 36, 37 and 38th of Euclid including alſo, in a Corollary, the 39th and 40th.

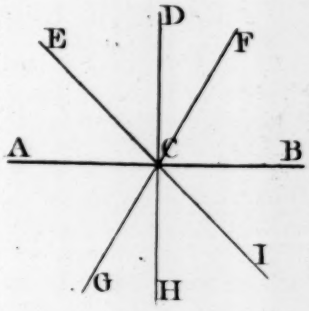
This property of Parallelograms and Triangles is, undoubtedly, very extenſive; but it is all included in the firſt part of the 18th (i. e. in the 35th of Euclid) having previouſly demonſtrated, in the 17th, that every Triangle is equal to half a Parallelogram, having the ſame Baſe and Altitude; which, Euclid deduces and demonſtrates from the other. To dwell ſo long on one Property, in ſix Propoſitions, becomes tedious, if not trifling; for where is the difference, whether Parallelograms, or Triangles, have the ſame or an equal Baſe?

The alterations, which I have made in the firſt Book, are very conſiderable; and the manner of demonſtrating is, in geaeral, different, and more conciſe; as to the clearneſs of it, it muſt be left for others to determine. I do not doubt but the judicious, candid, and impartial Reader will concur with me, or not diſapprove, entirely, the liberty I have taken. I thought it incumbent on me to advertiſe him of the alterations, and, by pointing them out, he may, with more eaſe, form a judgment of them.

THE

THE firſt ſix Propoſitions, and the Corollaries deduced from them, contain all the properties of Right Lines, and Angles formed by their Interſections; which are eſſentially neceſſary to be known, before we can obtain a knowledge of the properties of Plane Figures. But, I think it ſcarce worth the loſs of time to demonſtrate them, they being in a manner ſelf evident; inſomuch, that the greateſt part of them may be, and are, by ſome, given as firſt Principles, or ſelf evident Propoſitions.

1. For, ſince (by Def. 10.) one Right Line ſtanding perpendicularly on another makes the Angle on each ſide equal to the other, they are, therefore, Right Angles.



The Perpendicular, CD, being common to both Angles, ACD and DCB; the two other Sides AC, CB are, conſequently, in one Right Line, ACB. And, it is clearly evident, that any other Line, or number of Lines, as CE, CD, CF, at the ſame Point C, muſt make all the Angles, ACE+ECD+DCF+FCB, equal to the two Right Angles (ACD+DCB.)

Hence, the firſt Propoſition and its Corollaries are evidently clear and manifeſt.

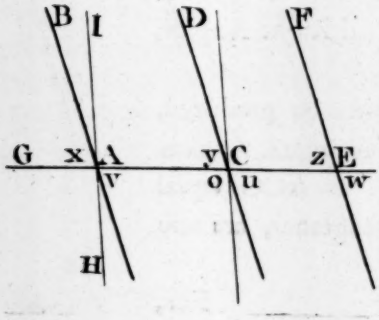
2. Again; (by Def. 11.) if the Perpendicular, CD, be produced, towards H, there is generated equal Angles, to the oppoſite, on either Side, for, they are all Right Angles, conſequently equal, by the 9th Axiom.

So likewiſe, if any other, inclined, Line be produced, as EC, or FC, it will, neceſſarily, produce equal Angles, to the oppoſite, on the other ſide of AB; as ACG equal FCB, and GCB=ACF; and which, together, are alſo equal to two Right Angles.

For, if EC biſects the Right Angle ACD, CI will alſo biſect the oppoſite, HCB; and, if CF triſect the Right Angle DCB; the oppoſite Angle, ACH, will alſo be triſected, if FC be produced, towards G; and ſo, in whatever proportion DCB is divided, by FC, the continuation of FC will, neceſſarily, divide the oppoſite, ACH, in the ſame Proportion; wherefore, the Ang. GCH=DCF, and ACG=FCB; alſo, GCH+HCB=DCF+ACD; and ACG+GCB=FCB+ACF; i. e. they are each equal to two Right Angles; and conſequently, all the Angles about the Point C are equal to four Right Angles.

Hence, it is plain, that the firſt and ſecond Propoſitions, with their Corollaries, are clearly deducible from the 10th and 11th Definitions; after what has been ſaid in the Theory of Plane Angles is well underſtood.

3. It is alſo, I preſume, eaſy to be conceived, and which I think no Perſon, at leaſt, who has any Talent for Geometry) can be at a loſs to conceive; that, two or more parallel Lines muſt neceſſarily have the ſame Inclination to any Right Line which cuts them all; i. e. the Angle of Inclination, with each Line, is equal to one another; the bare Idea of Parallelism ſeems to indicate it; which being known, or underſtood, all the reſt follow of courſe, viz. that the Alternate Angles are equal to one another, and the two internal Angles, on each Side, are equal to two Right Angles.



For, ſuppoſe the Right Lines AB, CD, and EF to be parallel amongſt themſelves, and all cut by any Right Line, GE; making, with them, the Angles x, y, z; and, ſuppoſe AB to be moved, directly, to CD, ſtill keeping parallel to CD, it muſt neceſſarily coincide with CD; and, conſequently, the

the Angle GAB will always remain the ſame, and, at laſt, coincide with ACD.

And if AB be moved directly forward, to EF, ſtill keeping parallel to its firſt poſition, it will alſo coincide with EF, and the Angle GAB, or ACD, with CEF.

Hence, it is plain that the Angles x, y and z, made with GE, are all equal to one another; and, being equal to one another, the Right Lines AB, CD and EF are conſequently parallel between themſelves.

4. Again; ſince the Angles x, y and z are equal to one another; and the oppoſite Angles, v, u, w, are alſo equal between themſelves, and to the other (as it was made evident in the foregoing) conſequently, the Alternate Angles, v and y, u and z, &c. are equal; for, they are all equal to one another, as above (the Lines AB, CD and EF being parallel) and, converſely, if the Alternate Angles are found to be equal, the Lines, forming them, are conſequently parallel.

5. Alſo; ſince a Right Line cuting another Right Line makes the Angles, on both Sides, equal to two Right Angles (by the firſt) it is evident, that, if the Angles x, y and z are equal to one another, and the adjoining Angles x+A and y+C, &c. are each equal to two Right Angles; conſequently, A, C, and E are alſo equal between themſelves, and the internal Angles (between each two Lines) A added to y, and C added to z, are alſo, each, equal to two Right Angles; by the 6th. Axiom.

6. And, becauſe A+v and y+o are each equal to two Right Angles; and, A+y, on one Side GE=two Right Angles; conſ. v+o, on the other Side GE, are, alſo, equal to two Right Angles. Therefore, if two parallel Right Lines, AB & CD, &c. are cut by any other Right Line, GF, the internal Angles, on each Side, are equal to two Right

Angles; and, conſequently, if a Right Line cuts two Right Lines, which are not parallel (as CD and HI) they will meet on that Side where the two internal Angles (DCA+CAI) on the ſame Side, are leſs than two Right Angles; which, is Euclid's 12th. Axiom.

Now ſince all theſe properties of Right Lines, and of Angles formed by Right Lines, are manifeſt, from what has been advanced already, there is, I think, but little occaſion for Demonſtration; nevertheleſs, in conformity to the Antients, I have formed them into Propoſitions, for the ſatisfaction of the ſcrupulous, and for the ſake of reference, in other Works, as well as in this; yet, I muſt own, I am clearly of opinion, that this brief extract is ſufficient, towards attaining all the neceſſary knowledge inculcated by the firſt ſix Propoſitions. It will, at leaſt, be of uſe to the young Student, in giving him a clear Idea of the Properties therein demonſtrated; which, by being more familiarized to them, will appear more evident and conſpicuous; the neceſſary knowledge, contained in them, will be deeper rooted and more ſecurely eſtabliſhed, and the Pupil better prepared to proceed with the Demonſtrations of the properties of Right lined Figures.

An explanation of the Notes and Abbreviations made uſe of in this and in other mathematical Treatices.

$=$ Note of Equality.
Thus; $A=B$, ſignifies that A and B are equal; and is thus read, A is equal to B; frequently, A equal B.

$+$ Note of Addition, Plus, or more.
Thus $A+B$, ſignifies the Sum of A added to B, A and B repreſenting any Quantity, whatever. Let A be 5, and B, 3; then $A+B=8$.

$-$ Note of Subtraction; Minus, or leſs.
Thus $A-B$, ſignifies the ſum of B taken from A, $A-B=2$, muſt be read, A, leſs B, is equal to 2.

$\times$ Note of Multiplication.
Thus $A\times B$, ſignifies, A multiplied into B.
$A\times B=15$. A multiplied into B is equal 15.

$\div$ Note of Diviſion.
Thus, $A\div B$ ſignifies, A divided by B.
Let A be 6 and B 2; then, $A\div B=3$.

$::::$ Note of Equality of Proportion.
Thus, $A:B::C:D$, ſignifies, that A bears the ſame Proportion to B, as C to D. $6:9::2:3$.

$\div\!\!\div$ Note of continued Proportion, or geometrical Progreſſion.
Thus, $A:B:C:D\div\!\!\div$ ſignifies, that A has the ſame Proportion to B, as B has to C, and, as C to D. In Numbers, thus, $1:3:9:27\div\!\!\div$

$\square$ Square. Thus, $AB\square$, ſignifies, the Square of the Line AB. A——B.

$\overset{A}{D}\ \overset{B}{\square}\ \overset{}{C}$ Rectangle. Thus, $AC\square$; or ABC, ſignifies the Rectangle under the Lines AB and BC; or, $AB\times BC$. And, $ABC\square=BD\square$, muſt be read thus; the Rect. ABC is equal to BD Square, or to the Square of BD.

Rect-

Rectangles and other Parallelograms, are frequently denoted by two Letters, at oppoſite Angles. As AC or DB.

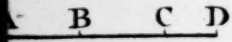
2ABC; or 2AB□; muſt be read, two Rectangles ABC, or twice the Rect. AB; and, 2 or 3 AB□ Square muſt be underſtood, twice or thrice the Square of AB.

△ Triangle. Thus, △ABC, ſignifies the Triangle ABC.

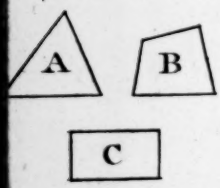
AXIOMS.

An Axiom (as already defined) is a manifeſt Truth, clear in itſelf, and therefore, does not require to be demonſtrated: Nevertheleſs, an Illuſtration will not, I preſume, be thought ſuperfluous or unneceſſary.

1. Things which correſpond, coincide, or mutually agree with each other, in every Part, are equal.
And if the wholes be equal, the halves are alſo equal.

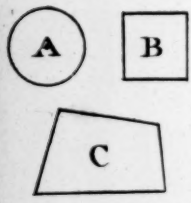


Thus, A is equal B; if every part of the Figure A coincides with B.



2. The whole is greater than a Part of the ſame Thing. For it is equal to all its Parts.

Thus, AD=AB+BC+CD.



3. If two, or more, Things, or Quantities, are each equal to the ſame third Thing or Quantity, they are equal between themſelves.

Thus, if A=C, and if B=C, then A=B.

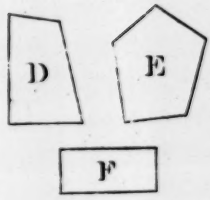
4. Things, or Quantities, which are each equal to half (or any other equal portion) of the ſame third Thing, are equal to one another.

Thus, if A and B are each equal to half of C, A and B are equal.

5. Things,

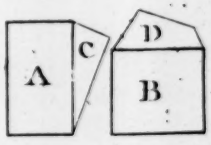
5. Things, or Quantities, which are each double, or triple, &c. of the ſame third Thing, or Quantity, are equal to one another.

Thus, if D and E are each equal twice or three times F, D and E are equal.



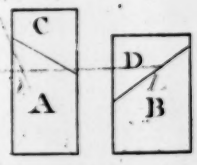
6. If to equal Things be added equals, the Sums are equal.

If to the equal Rectangles A and B, be added equals, C and D; then, C + A = D + B.



7. If from equal Things be taken away equals, the remainders will be equal.

If from the equal Rectangles, A and B, be taken away equals, C and D, the remainders of A and B are equal.

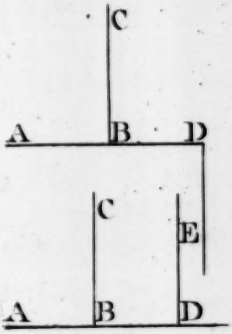


8. If to or from unequal Things be added or taken away equals, the Sums or Remainders will be unequal.

This neceſſarily follows from the two laſt.

9. All Right Angles are equal to one another.

The Right Angle ABC = the Right Angle BDE.



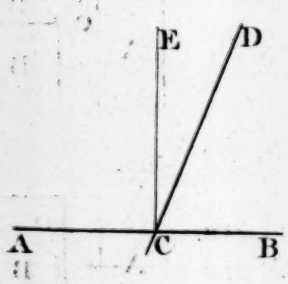
10. Two or more Right Lines, being perpendicular to the ſame Right Line, and being all in the ſame Plane, are parallel. BC is parallel to DE.

The Angles ABC, BDE, being Right Angles, are equal; by the laſt.

THEO-

THEOREM I. 13. Euclid.

If one Right Line touch or cut another Right Line, at any Point apart from its Extremes, it will make two Angles; which are either both Right Angles, or, both together, equal to two Right Angles.



Let DC be a Right Line, touching AB in the Point C, making two Angles, ACD, DCB; I ſay, they are either Right Angles, or both together equal to two Right Angles.

If CD be perpendicular to AB the thing is manifeſt; by Def. 10.

But if CD does not make Right Angles with AB; draw CE perpendicular to AB.

DEM. Now, ACE, ECB are Rt. Angles (Con.) th. equal.

But, ACD=ACE+ECD; alſo DCB=ECB−ECD.

Therefore, ECD, the exceſs of the Obtuſe Angle, ACD, to the Right Angle ACE, is alſo the deficiency of the acute Angle DCB to the Right Angle ECB.

Conſ. ACD+DCB is equal to ACE+ECB. - Ax. 2.

COR. 1. Hence, if from the ſame Point in a Right Line (as C in the Line CD) two Right Lines AC, CB are drawn, on contrary Sides and in the ſame Plane, making Right Angles, or Angles equal to two Right Angles, with the given Line, they will conſtitute one Right Line. ACB.

2. Two Angles at the ſame Point, and on the ſame Side of a Right Line, being equal, they are conſequently Right ones.

Wherefore, when two Angles are contiguous or adjoinin, if one of them ACE, be a Right Angle, the other ECB is alſo a Right one; and if one of them ACD be Obtuſe, the other, DCB, is conſequently Acute.

3. Two

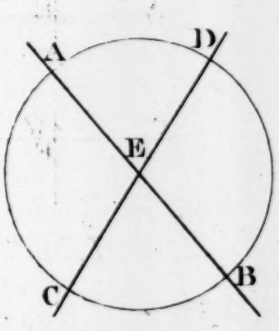
3. Two or more Right Lines meeting in the ſame Point of another Right Line, make all the Angles, together, equal to two Rt. Angles. ACE + ECD + DCB = ACE + ECB.

4. When one Right Line makes Angles with another, if one of the Angles be known the other is alſo known.

For, it is the difference between the Angle known and two Right ones. As, ACD = ACE + ECB − DCB.

THEOREM II. 15. Euclid.

Vertical or oppoſite Angles, made by the Interſection of two Right Lines, are equal to one another.



Let AB and CD interſect in E; then, the Angles AEC, BED, are equal; alſo, AED is eq. to CEB.

Dem. The Right Line AE, touching the Right Line CD, makes the Angles AEC, AED, together, equal to two Right Angles. - Th. 1.

And, becauſe DE touches the Right Line AB, the Angles AED, DEB, are alſo equal to two Right ones. - - - - - - - - - ſame.

Therefore, the Ang. AEC + AED = AED + DEB. - Ax. 9.

and if there be taken away the common Angle AED, there will be left, the Ang. AEC equal to DEB. - Ax. 7.

In the ſame manner, AED may be proved equal to CEB.

Or, it may be demonſtrated after another manner, thus.

About E, the point of Interſection, as a Center, deſcribe a Circle (with any Radius at diſcretion) cuting the two Lines in A, C, B, and D. Then, the Ark AD + DB = CB + BD.

For, they are, each two, equal to the Semi-circumference of a Circle. - - - - - - - - See N. B. Def. 21.

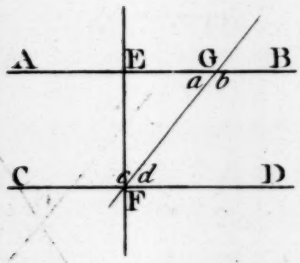
Wh. taking away the Ark DB, there remains AD = CB; and, the Angles AED, CEB, having equal Arks, are equal.

Cor. Hence, if two Right Lines interſect each other, the Angles they make are equal to four Right Angles.

2. All the Angles which can be made about a Point are equal to four Right Angles.

THEOREM III.

If a Right Line, cuting two Right Lines, makes all the Angles, on both Sides, equal between themſelves, the two Lines are parallel.



Let the Right Line EF cut two Right Lines, AB and CD, making the Angles AEF, EFC, BEF, and EFD equal to one another; then AB is parall. to CD.

Dem. Becauſe the Angle AEF=BEF, EF is perpendicular to AB. - - - - - - - - - - - - - - Def. 10.
And, becauſe EFC=EFD, EF is, alſo, perpend. to CD.
Wherefore, ſince EF cuts both Lines, AB and CD, perpendicularly, the Angles AEF, EFC, &c. (being equal) are Right Angles; and, AE, EB, CF, and FD are perpendicular to EF. - - - - - - - - - - Def. 11.
But, AE, EB is the ſame with AB, & CF, FD, with CD.
Therefore, AB and CD, being both perpendicular to EF, are parallel. - - - - - - - - - - - Ax. 10.

Cor. From hence it is manifeſt, that, any Right Line, cuting two parallel Lines, makes the internal Angles, on each Side, equal to two Right Angles.

For,

For, if they are cut perpendicularly, as EF, it is clear.

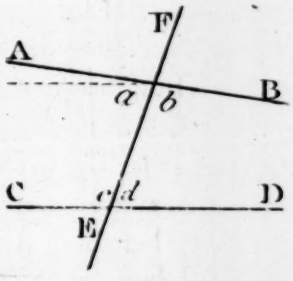
And, ſince any other Line (FG) cuting them both, makes the adjoining Angles (a and b, c and d) with each Line (AB and CD) equal to two Right Angles. - - - Th. 1. Conſ. all the internal Angles (a, b, c, and d) which it makes with them both, are equal to four Right Angles. - Ax. 2.

Now, ſince AB is parallel CD (Theo.) and the two internal Angles, on one Side EF, are equal to thoſe on the other Side (Hyp.) it is manifeſt, that the internal Angles, on one Side, are, alſo, equal to thoſe on the other Side, of any Right Line (FG) cuting them both; wh. $a+c=b+d$.

But, all the four internal Angles are equal to four Right Angles; conſequently, the internal Angles, $a+c$, and $b+d$, on each Side, are, each two, equal to two Right Angles.

Cor. 2. If two Right Lines, which are not parallel, be cut by another Right Line, they will meet on that Side where the internal Angles are leſs than two Right ones.

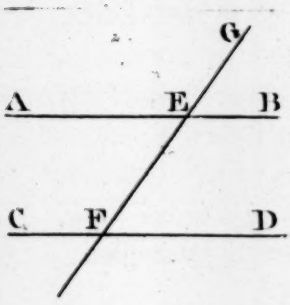
For, if $b+d$ be leſs than two R. Angles $a+c$ is greater; becauſe, $a+b+c+d=$ four Right Angles. - - Th. 1. Wherefore, AB will incline to CD, and they will, if produced, meet on that ſide EF, towards B and D, on which the Angles $b+d$ are leſs than two Right ones. Becauſe, the Angles, b and d, on that Side EF, are, together, leſs than $a+c$, on the other Side.



If this Theorem cannot be admitted, as perfect Demonſtration, it may be ſuppoſed a Lemma to the next; being an illuſtration, at leaſt, of Euclid's 12th Axiom; which, in my opinion can not be received as ſuch, by any Perſon as yet unacquainted with Geometry. From that conſideration, it may paſs with the moſt ſcrupulous. My deſign is to make Geometry underſtood, and intelligible to common Capacities, in the eaſieſt manner.

THEOREM IV. 27. Euclid.

If a Right Line, cuting two Right Lines, makes the Alternate Angles equal, to one another; or, ſhall make the external Angles equal to the internal and oppoſite; the two Lines are parallel.



Let FG be a Right Line, cuting the two Right Lines AB and CD, and making the Angles AEF, EFD, equal to one another; then, the two Lines, AB and CD, are parallel.

For, if the Angles AEF, EFC, be not equal to two Right Angles, they are either greater or leſs; ſuppoſe them greater; then, the Lines, AB and CD, will meet on the other Side, towards B and D. - - - - - - - - - - C. 2. 3.

Now, the Angles BEF, EFD, are leſs than AEF + EFC.
And the Ang. AEF + FEB = EFC + EFD, - - Ax. 9.
for, they are, each two, equal to two Right Angles.-Th. 1.
But, the Ang. AEF = EFD (Hyp.) wh. BEF = EFC.-Ax. 7.
Conſequently, BEF + EFD = EFC + EFD;
and, BEF + EFD = AEF + EFC, - - - - - - Ax. 6
i. e. they are, on each ſide, equal to two Right Angles.
Therefore, AB is parallel to CD. - - - Th. 3. and Cor.

2. Let the Angle GEB be equal to EFD.

Now, the Ang. EFD = GEB (Hyp.) and, AEF = GEB. 2.
Wherefore, AEF is equal to EFD. - - - - - Ax. 3.
But, the Angles AEF, EFD, are Alternate; and, it is already proved, that, if Alternate Angles be equal the Lines are parallel. Therefore AB is parallel to CD.

COR.

Cor. 1. Conv. If a Right Line cuts two Right Lines, which are parallel, the Alternate Angles are equal to one another.

For, AEF+BEF are equal to two Right Angles. Th. 1. Alſo, BEF+EFD are equal to two Rt. Angles. C. 1. 3. Wherefore, AEF=EFD (BEF being common) Ax. 3. and conſeqnently, BEF=EFC; by the ſame.

Cor. 2. If a Right Line cuts two parallel Lines, it makes equal Angles with them both. viz. GEB equal to GFD.

N. B. From this Theorem, is deduced the fifth Problem; according to the firſt Method, there preſcribed.

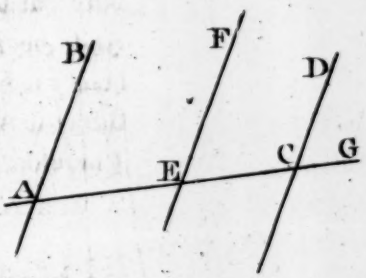
THEOREM V. 30. Euclid.

If two, or more, Right Lines are parallel to any other Right Line, they are parallel between themſelves.

Let the two Right Lines, AB and CD, be both parallel to the ſame Right Line, EF.

I ſay, that AB is parallel to CD.

Let any Right Line, (AG) cut them all; being produced, if neceſſary



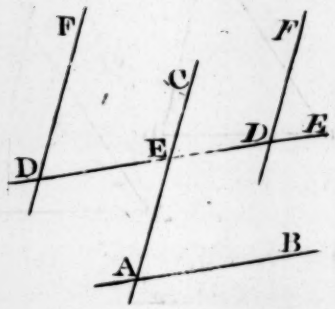
Dem. Becauſe AB is parallel to EF, and they are both cut by the Rt. Line AG, the Angle BAE=FEC. -- C. 2. 4. And, for the ſame reaſon, the Angle DCG=FEC; wherefore, the Angle BAE is equal to DCG. - Ax. 3. But, thoſe Angles are internal and oppoſite. - Def. 16. therefore, AB is parallel to CD. - - - 2. Th. 4.

SCHOL. *This Theorem might well paſs for an Axiom; for, by adopting parallelism for Things, or Quantity, 'tis the ſame as Axiom the third.*

THEO-

THEOREM VI.

If two Right Lines cut each other, and, if there be two other Right Lines (in the ſame Plane with the former) parallel to them, reſpectively, and alſo cut each other, they contain equal Angles.



Let the Right Lines DE and DF (or *DF*) be reſpectively parallel to AB and AC.

I ſay, the Angle FDE is equal to CAB.

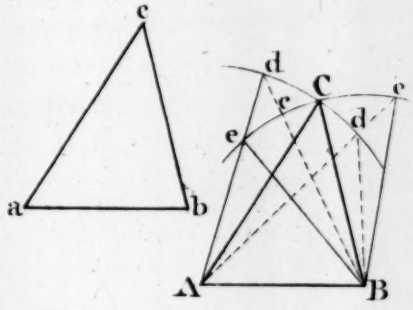
Produce either Side (if it be neceſſary) as DE (or *ED*) cuting AC, at E.

DEM. Now, becauſe DE is parallel to AB, and they are both cut by AC, the Angle CAB=CE*D*. - - C. 2. 4. And, becauſe DF (or *DF*) is parallel to AC, and are both cut by DE, the Angle FDE=CE*D*. - - - - ſame. But, the Angle CAB=CE*D*; and FDE=CE*D*. proved. Therefore, the Angle FDE is equal to CAB. - - Ax. 3.

This valuable Theorem is not in Euclid, except the 10*th of the eleventh Book, may be compared with it; in which, the two Lines (cuting each other) are, reſpectively, in different Planes; but they are, neceſſarily, either in the ſame Plane, or in parallel Planes, and holds equally true.*

THEOREM VII. 8. Euclid.

If the three Sides of one Triangle are equal, reſpectively, to the three Sides of another Triangle, they are alſo equiangular; i. e. the Angles oppoſite to equal Sides, are alſo equal, and the whole Triangles are congruous, or equal, each to the other.



Let the Triangles a b c, ABC, have all their three Sides equal; viz. a b equal to AB, b c equal BC, and a c equal AC.

I ſay, the Angle a is alſo equal to A, b equal B, and c equal C; and the Triangle, a b c, equal to ABC.

Dem. Suppoſe the Triangle a b c applied to ABC; any Side, a b, to its equal AB. - Poſt. 5.

Then, if the Triangles are alike ſituated, a c will fall on AC, and b c on BC; the Angle a will coincide with A, b with B, and c with C.

For, becauſe the Side a b is equal to AB, the Extremes, a and b, will fall on A and B; and, becauſe a c is equal to AC, and b c to BC, the Point c muſt neceſſarily fall on C

If not, let c, if poſſible, fall at d, in the Ark C d, deſcribed on A, with the radius a c; wh. A d = AC, equal a c.

But, B d is not equal to b c; for, if an Ark of a Circle be deſcribed, on B, with the Radius b c, being equal to BC, it will cut the Ark, d d, at C, and the Line A d, at e. Wherefore, the Point c cannot fall at d, nor elſewhere but on C, in the interſecting Point of the two Arks.

Th. the Triangles, agreeing in every part, are congruous.

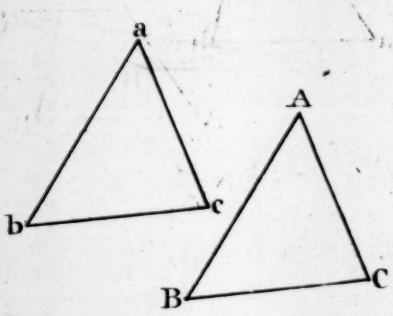
N. B. This Property does not hold good in any other Figure.

Hence, an Angle is made equal to a given Angle, as in Pr. 4; and hence, an Angle is biſected according to the 2nd method, Prob. 9th; viz by making two Triangles, congruous with each other,

THEO-

THEOREM VIII. 4. Euclid.

Triangles having two Sides, in each, reſpectively, equal, and if the Angles, contained by thoſe Sides, are alſo equal; then, the remaining Side, of the one, ſhall be equal to the remaining Side of the other; the remaining Angles, of the one, equal to the remaining Angles of the other, viz. thoſe that are oppoſite equal Sides; and the whole Triangles are equal, and congruous.



Let the Triangle, a b c, have two Sides, a b, b c, equal, reſpectively, to AB, and BC, of the Triangle ABC; and, let the Angle a b c be equal to ABC.

Then, the Side a c is equal to AC; the Angle a is equal A, and c equal C; and the whole Triangle, a b c, equal to ABC.

DEM. For, ſuppoſe the Triangle, a b c, applied to, or laid upon, the Triangle ABC, in ſuch wiſe, that the Side a b, of the one, agrees with AB of the other Triangle; the Extreme a, with A, and b with B. - - Poſt. 5.

Then, the Side b c will fall upon BC; - by Def. 14; for, they contain equal Angles, by the Hypotheſis.

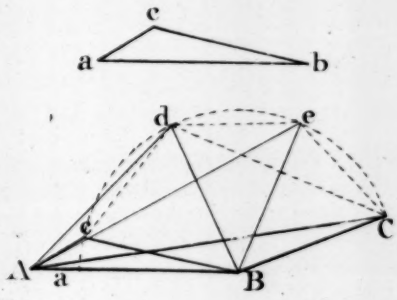
The Extreme c, of the one, will coincide with C, of the other; becauſe, b c is equal to BC.

Therefore, the remaining Sides, a c and AC, muſt neceſſarily agree, and be equal to each other; the Angle a will coincide with A, and c with C; and conſequently, the whole Triangles, a b c, ABC, are congruous; therefore, the Triangle a b c is equal to ABC.

COR.

Cor. Hence, if two Sides of one Triangle be equal, reſpectively, to two Sides of another Triangle, and the contained Angle, of one, be greater or leſs than the contained Angle of the other, the remaining Side will alſo be greater or leſs; and converſely.

For, ſuppoſe two Triangles, a b c, ABC, having their Sides, a b and b c, reſpectively equal, to AB and BC.



It is evident, that, if the Angle a b c was the leaſt poſſible; i. e. if the inclination of the Sides, a b, b c, was ſuch, that it is not poſſible to conceive a leſs Angle, the two Sides, a b, b c, would nearly coincide; and conſequently, the length of the remaining Side, a c, would be ſomewhat more than the difference between a b and b c; viz. a c = a b − b c, nearly.

Again. Suppoſe the Angle made by the Sides AB, BC, equal a b, b c, the largeſt poſſible, ſo as, nearly to fall into one Right Line; it is manifeſt, that the remaining Side, AC, would be ſomewhat leſs than the length of both, AB and BC; i. e. AC = AB + BC, nearly.

Hence it is plain, that, whether the Angle, a b c or ABC, contained between a b and b c, or AB and BC, be greater or leſs, the remaining Side, a c or AC, will alſo be greater or leſs; and converſely, the greater or leſs the Side, a c or AC, the greater or leſs is the Angle, a b c or ABC.

For, ſuppoſe the Triangle a b c applied to ABC, the Side a b to AB, the Angle a to A, and b to B; and let the Ark of a Circle be deſcribed, from c to C, on the Center B.

Then, ſuppoſe the Angle a B c opened or enlarged, to the ſeveral Apertures, a B d, a B e, a B C; it is evident, that as the Angle a B c is increaſed, at a B d, &c. ſo is the remaining Side, A d, A e, &c. continually longer; from A c, the ſhorteſt poſſible, to AC, the longeſt.

Hence the 24th and 25th of Euclid are clear, and manifeſt.

What I have here advanced is, I preſume, ſufficient conviction, if not Demonſtration, of the properties therein contained.

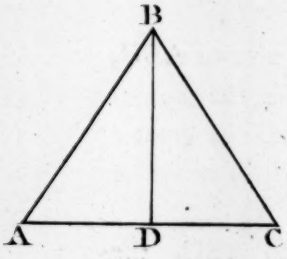
The ſhorteſt way to knowledge is the beſt; where perfect Demonſtration is neceſſary, I ſhall be as brief as is conſiſtent with perſpicuity, and where it is not, I ſhall briefly illuſtrate.

Thoſe who are not ſatisfied with this Demonſtration of the two laſt, ſelf evident, Propoſitions (of Euclid) may refer back, after the 12th (and this is not of uſe between) and, by drawing the Right Lines de, eC, &c. they will find, that the Side Ae, or AC oppoſite to an obtuſe Angle, Ade, or AeC, is greater than the Side Ad, or Ae, which is oppoſite to an acute Angle, Aed, or ACe.

The 7th might be a Corollary to the 8th. For ſince, in two Triangles, if there be found two Sides, in one, equal reſpectively to two Sides in the other, and containing equal Angles, the third Sides are equal; it certainly follows, that if all the three Sides are reſpectively equal, the Triangles are equal and congruous; ſeeing it is not poſſible for an Angle to be greater or leſs, but the Side oppoſite is alſo greater or leſs.

THEOREM IX. 5. Euclid.

In Iſoſceles Triangles, the Angles at the Baſe are equal to one another, i. e. if a Triangle have two equal Sides, the Angles, oppoſite to them, are alſo equal.



Let ABC be an Iſoſceles Triangle; the Side AB equal to BC.

Then, the Angle A is equal to C.

Biſect the Angle ABC (Pr. 9.) made by the equal Legs, AB, BC, and draw BD; it will alſo biſect the Baſe, AC.

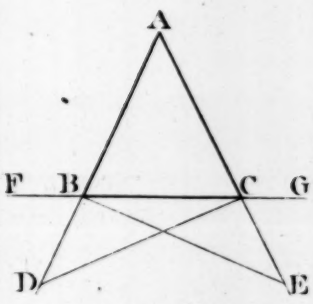
DEM. In the Triangles ABD, DBC, the Sides AB, BC are equal (Hyp.) and, BD is common to them both.

Now, the Sides AB, BD are = CB, BD, reſpectively.
And the Angle ABD is equal to DBC. - - - Con.
Th. AD=DC, and the Triangle ABD=DBC. - 8.
But, in congruous Triangles, oppoſite equal Sides are equal Angles. - - - - - - - - - Th. 7 and 8.
Th. the Ang. A=C, being oppoſite the common Side BD.

Or

Or it may be demonſtrated thus, after Euclid.

Let the equal Sides, AB, AC, be produced below the Baſe, to D and E.
Make AD eq. AE, and join DC and BE.



Then, in the Triangles ABE, ACD, there are two Sides, AB, AE, reſpectively equal to AC, AD; and the Angle A is common; wherefore, DC=BE; and the Angle ABE is equal to ACD. - - 8.

But, BD=CE (Ax. 7.) and BC being common; the Triangle, and conſequently, the Angle, DBC=BCE. - 7.
Alſo, the Angle BCD=CBE; conſ. ABC=ACB. - Ax. 7.

Cor. 1. The external Angles, made by producing equal Sides of a Triangle, or the Side which lies between them, are alſo equal, amongſt themſelves.

For, the Ang. DBC=BCE; conſ. ABF=ACG. - 2.

2. An Equilateral Triangle, is alſo equiangular.
For, oppoſite to equal Sides are equal Angles.

3. If a Triangle have two equal Angles, the Sides oppoſite them are alſo equal. And, if all the three Angles of a Triangle are equal, it is equilateral.

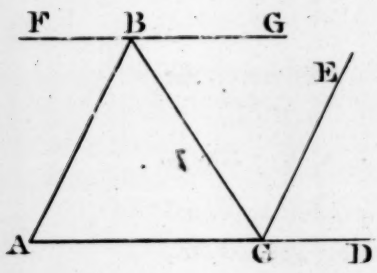
4. A Right Line, biſecting the Angle, and the Baſe, contained between equal Sides, is perpendicular to the Baſe; and conſequently, two Right Lines drawn from the ſame Point in a Perpendicular to a Right Line, to Points, equally diſtant from the Perpendicular, in that Line, are equal.

For, the Angles ADB, BDC, being equal, are conſequently Right Angles. (See firſt Figure.) C. 2. Th. 1.

N. B. Hence are performed the 6th, 7th and 8th Problems; viz. by conſtructing Iſoſceles Triangles.

THEOREM X. 32. Euclid.

If any Side of a Triangle be produced, the external Angle is equal to the two remote Angles of the Triangle. And, the three Angles of every Triangle are, together, equal to two Right Angles.



In the Triangle ABC, let any Side, as AC, be produced. Then, the Ang. BCD is equal to the two Angles, A and B.

From C, draw CE parallel to AB.

Because CE is parallel to AB - Con. and they are both cut by the Right Line BC, the Angle BCE=ABC. - Th. 4. And, because AB and CE, being parallel, are cut by the Right Line AD, the external Angle, ECD=BAC, the internal and opposite; - - - - - - - - 2. Th. 4. But, the Ang. BCE (eq. B) + ECD (eq. A)=BCD. Ax. 2. Therefore, the external Angle, BCD, = A + B, the two remote Angles of the Triangles. Q. E. D. - Ax. 3.

2. The three Angles, A + B + BCA, are equal to two Right Angles.

Having produced AC, and drawn CE, parallel to AB.

Dem. The Angle BCE=ABC, and ECD=BAC - 4. Wh. BCD (eq. BCE + ECD) is equal to ABC + BAC. But, the Angle ACB, + BCD = two Rt. Angles. - - 1. Therefore, the three Angles, A + B + BCA, are equal to two Right Angles.

This 2nd part is also elegantly proved, by drawing a Right Line, FG, through any Angle, B, parallel to the opposite Side, AC.

Dem.

DEM. For, the Angle FBA = A, and GBC = BCA - 4, conſ. the Ang. A + ABC + BCA (eq. FBA + ABC + CBG) are equal to two Right Angles. - - - C. 3. Th. 1.

COR. 1. Hence, if one Angle of a Triangle be equal to the other two Angles, it is a Right one.

For, if either Side, containing the Right Angle, be produced; the external Angle, being equal to the two remote ones, is equal to the Right Angle to which it is contiguous. (Defin. 11.)

2. There can be but one Right Angle, or one Obtuſe Angle, in a Triangle.

Becauſe, if there be one Right Angle, the other two are, together, equal to a Right Angle.

3. If a Triangle have one Right Angle, the other two are acute, and equal to a right one; and, if the Triangle be Iſoſceles, the two Angles, at the Hypothenuſe, are each equal half a Right Angle. For, they are equal. - Th. 9.

4. If the ſum of two Angles of a Triangle be known, the other is alſo known.

For, it is the Complement of two Right Angles.

Alſo, if one Angle be known, the Sum of the other two is known.

5. If the Sum of two Angles of a Triangle, either together or ſeparately, be equal to two Angles of another Triangle, the two remaining Angles are equal.

For, the Sum of all the Angles of every Triangle is the ſame; i. e. they are equal to two Right Angles.

6. The equal Angles of every Iſoſceles Triangle are acute.

For, however ſmall the Angle at the Vertex, the Angles at the Baſe are, each, equal to half the difference between that Angle and two Right Angles.

7. The

7. The Angles of an Equilateral Triangle are each 60 Degs.

For, it is two thirds of a Right Angle, or one third of two Right Angles.

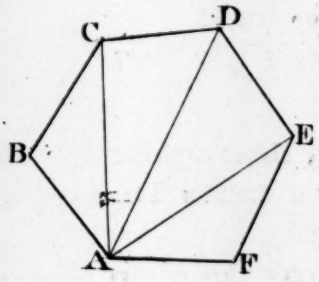
8. All the Angles of every Quadrilateral are, together, equal to four Right Angles.

For, by drawing a Diagonal, it is divided into two Triangles; and all the Angles, of each Triangle, are equal to two Right Angles; conſequently, the ſum of all the Angles, of both, are equal to four Right Angles; and they are equal to all the Angles of the Quadrilateral.

Note. The two following Theorems are alſo deducible from the laſt Propoſition; which, not being elementary, are not numbered amongſt the other Theorems.

THEO. 1. All the internal Angles, of every right lined Figure, are equal to twice as many Right Angles, wanting four, as the Figure has Sides.

Every right lined Figure, may be reſolved into as many Triangles, as the Figure has Sides, wanting two; by drawing Diagonals, from any Angle, to all the oppoſite Angles.



In the irregular Hexagon ABCDEF, let there be drawn the Diagonals AC, AD, and AE; which divides it into four Triangles, ABC, ACD, &c.

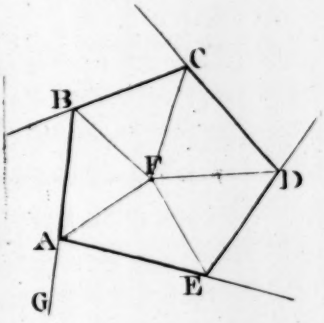
DEM. Becauſe, by the Theorem, the three Angles of every Triangle are equal to two Right Angles, the Sum of all the Angles of the four Triangles, ABC, ACD, &c. are equal to eight Right Angles.

And, they are equal to all the internal Angles of the Hexagon. Therefore, all the Angles of the Hexagon are equal to eight Right Angles.

But, a Hexagon has ſix Sides, and twice ſix is twelve; by taking away four, there will remain eight Right Angles; which are equal to all the Angles of the Figure.

Or;

Or; by aſſuming any Point, as F, within the Figure, ABCDE, and drawing Lines to every Angle, from that Point, FA, FB, &c. it will be divided into as many Triangles, as the Figure has Sides.



Now, the Angles of all the five Triangles are equal to ten Right Angles. - - Theo. 10.

But, one Angle, of each Triangle, is at the point F, within the Figure; and all the Angles about a Point are equal to four Right Angles (Cor. 2. 2.) conſequently, the remaining Angles, of the Triangles, are equal to all the Angles of the Figure; which, in a Pentagon, are equal to ſix Right Angles.

Cor. Hence, the Sums of all the Angles, of every right lined Figure, having an equal number of Sides, are equal.

Theo. 2. All the external Angles, made by producing every Side, of any right lined Figure, are equal to four Right Angles.

For, ſince the internal and contiguous external Angle, BAE+BAG, &c. are equal to two Right Angles (Th. 1.) the Sum of all the internal and external Angles, together, are equal to twice as many Right Angles, as the Figure has Sides.

But, all the internal Angles are equal to as many, wanting four; conſequently, the external Angles are equal to four.

Cor. The ſums of all the external Angles (taken together) of every Right lined Figure, are equal.

SCHOL. *This property of right lined Figures is, at firſt, really ſurprizing; for, the external Angles of a Triangle, are equal to all the external Angles of any Poligon, whatever, and of any number of Sides.*

The general utility of the knowledge of this property of Triangles, in the laſt Theorem, is very extenſive. It is, perhaps, the moſt perfect Demonſtration that can be given; and we may conceive an Idea of its extenſiveneſs, from the Corollaries and Theorems already deduced from it.

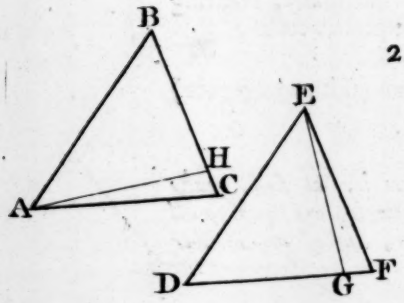
THEO-

THEOREM XI. 26. Euclid.

Triangles having two Angles, and one Side, reſpectively equal to two Angles, and a correſponding Side; whether it be the Side, between the equal Angles, or either of the other, which are oppoſite to equal Angles; the remaining Sides and Angles ſhall be equal, and the Triangles are congruous.

The firſt part of this Propoſition, viz. when the equal Sides lie between the equal Angles, is ſo very obvious, that it might have been made a Corollary to the 8th, from which it is eaſily deduced.

DEM. For, if AB, equal DE, be applied to DE (Poſt. 5.) if the Angle A be equal D, and B equal E (the equal Sides, AB, DE, lying between) then will AC fall on DF, and BC on EF. - - - - - - - - Def. 14. Conſ. the Angle C, will fall on F, and the Triangle ABC, coincide, or agree in every reſpect, with DEF.



2. Let ABC and DEF be two Triangles, whoſe Angles A and C, of the one, are, ſeparately, equal to D and F, of the other; and let the Side AB, of the one, be equal to DE, of the other.

Then, the Triangles, ABC, DEF, are equal, and congruous.

DEM. Firſt, if the Side AC be not equal to DF, it is either greater or leſs; ſuppoſe it leſs, and take DG equal AC, and draw EG.

Then,

Then, becauſe AB=DE (Hyp.) and DG=AC, by Suppoſition; the Sides, AB, AC, are reſpectively equal to DE, DG; and the Angle BAC=EDG. - - Hyp.
Wh. EG=BC and the Triangle DEG=ABC - - - 8.
conſequently, the remaining Angles, of the one, are equal to the remaining Angles of the other, which are oppoſite equal Sides; wherefore, the Angle DGE is equal ACB.
And, DFE=ACB (Hyp.) wh. DGE=DFE. - Ax. 3.
But, DGE is greater than DFE; - - - Th. 10.
for it is an external Angle of the Triangle FEG;
it is, therefore, both equal and greater, which is abſurd.
Therefore, AC is not leſs than DF.

After the ſame manner, it may be proved not greater; conſequently, AC is equal to DF.
Th. the Triangle ABC is equal to DEF; and congruous.

Again; if it be affirmed, that BC may be either greater or leſs than EF; ſuppoſe BC greater than EF.
Take BH equal EF, and join AH.

Then, the Angle BAC, being equal to EDF, is greater than the Angle BAH; which deſtroys the Hypotheſis.
Wherefore, BC cannot be greater or leſs than EF.

This Theorem concludes the general properties of Triangles, mentioned in the preamble to this Book; which I have kept as near together as poſſible; by which properties, peculiar to Triangles, only, we can, with certainty, affirm them to be congruous, conſequently equal.

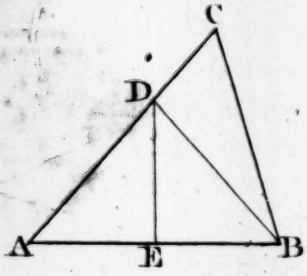
Firſt, when all the three Sides, of one Triangle, are equal, reſpectively, to the three Sides of another Triangle. (Th. 7.)

2nd. When there are found only two Sides reſpectively equal to two Sides, and containing equal Angles between thoſe Sides. (8.)

3dly. When two Angles, of the one, are reſpectively equal to two Angles of the other; and one Side of each Triangle, either lying between the equal Angles, or oppoſite to equal Angles, alſo equal; by this Theorem.

THEOREM XII. 18. Euclid.

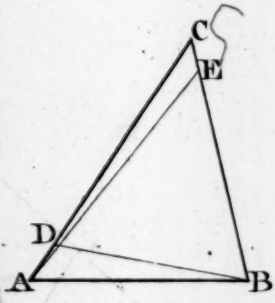
In every Triangle, the greateſt Side ſubtends the greateſt Angle; and the leaſt Side ſubtends the leaſt Angle.



Let ABC be a Triangle, whoſe Side AC, is greater than CB; I ſay, the Angle B, is greater than the Angle A.

Biſect the Side AB, in E; draw ED perpendicular to AB, and join DB.

DEM. In the Triangle ADB, the Side AD=DB; - C. 4 9.
wherefore, the Angle BAD is equal ABD. - - Th. 9.
But, the Ang. ABD (eq. BAD) is leſs than ABC. Def. 14.
Therefore, the Angle ABC (oppoſite to AC) is greater than BAC, which is oppoſite to a leſs Side, BC.



Or thus, after Euclid.

By Hypotheſis, AC is greater than CB.
Make CD equal to CB, and join DB.

Then, the Ang. CBD=CDB. - Th. 9.
But the Ang. CDB is greater than CAB. 10.
And, CBD (eq. CDB) is leſs than ABC.
Th. the Angle ABC is greater than CAB.

2nd. The leaſt Side ſubtends the leaſt Angle.

AB is leſs than BC; then, the Angle C is leſs than A.
Make BE equal BA, and draw AE.

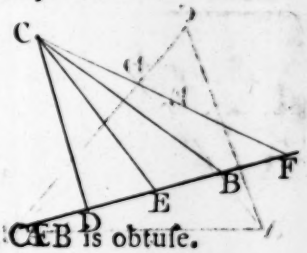
DEM. Now, the Angle AEB=BAE. - - - Th. 9.
But, AEB is greater than ACB. - - - - - - - 10.
Th. BAC, which is greater than BAE, is greater than ACB, which is leſs than AEB (equal EAB) Q.E.D.

COR.

Cor. 1. In Triangles, oppoſite the greateſt Angle is the greateſt Side; and the leaſt Side is oppoſite the leaſt Angle.

Cor. 2. A Perpendicular is the ſhorteſt Line, which can be drawn from any Point to a Right Line. And that Line, which falls neareſt to the Perpendicular, is ſhorter than the more remote ones.

Let CD be perpend. to AB. Draw, at pleaſure, CE & CB.

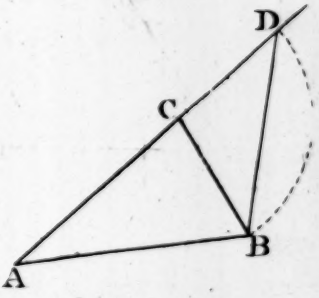


Now, ſince CDE is a Right Angle, CED is acute. - - - - C. 3. 10. Therefore, CE, which ſubtends the Right Angle, is greater than CD. - - Theo.

2ndly. Becauſe the Angle CED is acute, CEB is obtuſe. Conſequently, CB is greater than CE. - - - Theo. Therefore, CE, which is neareſt to the Perpendicular, CD, is leſs than CB, which is more remote from it.

THEOREM XIII. 20. Euclid.

In every Triangle, any two Sides, taken together, are greater than the remaining Side.



Let AB be the greateſt Side of the Triangle ABC.

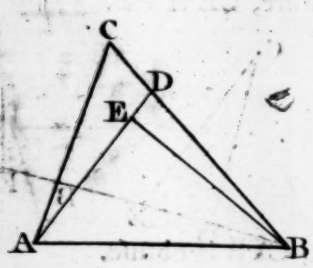
I ſay, the Sides AC, CB, are, together, greater than AB.

Produce AC to D; make CD equal CB, and join BD.

Dem. Then, becauſe CD = CB, the Ang. CBD = CDB - 9. But, the Angle ABD is greater than CBD, equal CDB. Then, ſince in the Triangle ADB, the Angle ABD is greater than ADB; the Side AD is greater than AB - 12. But AD = AC + CB (Con.) therefore, AC + CB, is greater than AB. Q. E. D.

THEOREM XIV. 21. Euclid.

If, from any Point within a Triangle, two Right Lines are drawn to the extremes of any Side; thoſe Lines, taken together, are leſs than the other two Sides of the Triangle, but they contain a greater Angle.



In the Triangle ABC, aſſume any Point, E, and draw EA, EB.

I ſay, that AE added to EB, is leſs than AC added to CB.

And, the Angle AEB is greater than ACB.

Produce AE to D.

DEM. In the Triangle ACD, the Sides, AC, CD, are greater than AD. - - - - - - - - - - - - 13.

Add DB to both; and AC+CD+DB, (eq. AC+CB) is greater than AD+DB. - - - - - - - Ax. 8.

Again. In the Triangle EDB, the Sides, ED, DB, are greater than EB.

Add AE to both; and DB+AD is greater than EB+AE.

But, AC+CB is greater than AD+DB, - - proved.

Therefore, AC+CB is ſtill greater than AE+EB.

2ndly. The Angle ADB is greater than ACB, and AEB is greater than ADB; - - - - - - - - 1. Th. 10.

conſequently, AEB is ſtill greater than ACB.

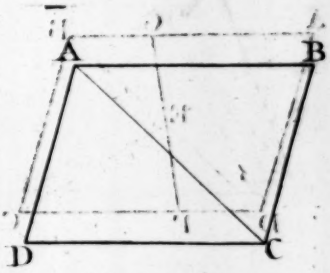
For, the Angle ADB is external, in reſpect of the Triangle ACD; and AEB, in reſpect of BED.

COR. Hence; if two Triangles have one Side, in each, equal to one another, or have one Side common; and, if the remaining Sides, of the one, be leſs, reſpectively, than the remaining Sides of the other, they will contain a greater Angle.

THEO-

THEOREM XV. 34. Euclid.

In every Parallelogram, the oppoſite Sides and Angles are equal; and, it is cut into two equal Triangles by a Diagonal.



Let ABCD be a Parallelogram.

The Side AB is equal to DC, and AD is equal to BC.

Alſo, the Angle D is equal to B, and A equal to C.

Draw either Diagonal; as AC.

Dem. By Def. 33d. AB is parallel to DC; and they are both cut by AC; wh. the Angle BAC=ACD. - Th. 4. And, becauſe AD is parallel to BC, and are both cut by AC; the Angle DAC=ACB; and AC is common; wherefore, the Triangles, ABC, ADC, are congruous-11. Therefore, the Side AB=DC, and AD=BC. - Def. 43. Alſo, the Angle B=D (Th. 11.) &, DAB=BCD-Ax.6. (For, BAC=ACD, and DAC=ACB (Th. 4.) as above; and conſequently, BAC+CAD=ACD+ACB.)

Cor. Hence, it is manifeſt, if Right Lines be drawn between the extreme Points of two Right Lines, which are equal and parallel (ſo as not to croſs each other) they are alſo equal and parallel.

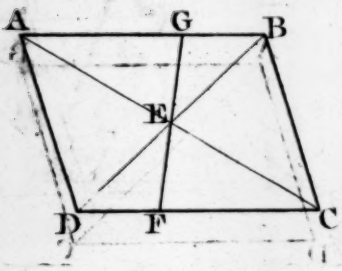
As AD and BC, joining the Points A & D, B & C.

Hence, the 33d of Euclid is evident.

THEO-

THEOREM XVI.

The two Diagonals of a Parallelogram biſect each other. And, every Diameter divides a Parallelogram into two equal and congruous Figures.



In the Parallelogram ABCD, draw the Diagonals, AC and BD.

I ſay the Diagonal AC is biſected by DB; and, DB is alſo biſected by AC, in the Point E, of their mutual Interſection.

Dem. Becauſe, AB is parallel to CD, and are both cut by AC and BD; the Ang. BAC=ACD, & ABD=BDC. 4. But, AB=DC (15.); wh. the Triangle AEB=DEC; and alſo congruous. - - - - - Th. 11. wh. DE=EB & AE=EC, being oppoſite equal Angles. Therefore, the Diagonals, AC, & BD, are biſected, in E.

2ndly. The Diameter, FG, divides the Parallelogram ABCD, into equal and congruous Figures. AGFD is equal to FGBC.

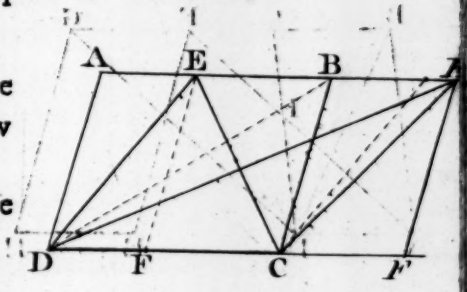
Dem. Now, the Triangle ABC is equal to ACD. - 15. The Angle AEG=FEC (2.) and GAE=ECF, - 4. and AE=EC (above) wh. the Tri. AGE=EFC. - 11. Let them be taken away; there remains AEFD=CEGB. and conſ. △AGE+AEFD=△EFC+CEGB. - Ax. 6. Th. the Trapezium AGFD is equal to FGBC. Q. E. D.

Cor. Every Right Line, which divides a Parallelogram, and paſſes through the Center, is biſected at the Center. As, FG is biſected in E.

THEO-

THEOREM XVII. 41. Euclid.

If a Triangle have the ſame Baſe as a Parallelogram and lie between the ſame Parallels, i. e. having the ſame Altitude, the Triangle is equal to half that Parallelogram.



In the Parallelogram ABCD, aſſume any Point, E, in the Side AB, and draw EC and ED.

The Triangle DEC is equal half the Parallelogram, ABCD.

Draw EF, parallel to AD and BC.

Dem. The Parallelograms, AEFD and FEBC, are biſected, by the Diags DE & EC; ADE=DEF, & FEC=EBC-15. Wh. the Triangle DEF+FEC=ADE+ECB. - Ax. 6. But, the Triangle DEC=DEF+FEC. - - Ax. 2. Therefore, △ DEC=ADE+ECB. Q. E. D.

Or; produce AB, make B*E* equal to AE, and draw C*E*.

Then, △ B*E*C (equal AED. 8.) compleats the Par. DE*E*C (15. Cor.) which is equally divided by the Diag. EC. The Triangle DEC=E*E*C (equal AED+ECB) - - 15.

Caſe the Second.

When *E* falls without the Par. ABCD, in AB produced. The Triangle D*EC* is equal half the Par. ABCD.

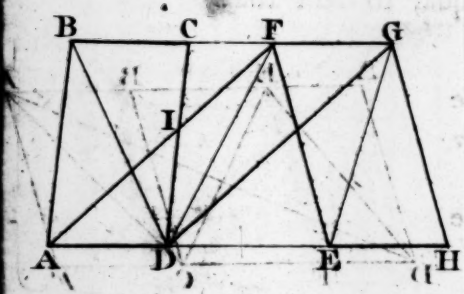
Draw *EF*, as before, parallel to BC; meeting the Baſe, DC, produced, in *F*

Dem. D*E* biſects the Par. A*EFD*; wh. A*ED*=D*EF*-15. and, the Diag. C*E*, cuts the Par. CB*EF*, in BC*E*=*ECF*. Wherefore, the Triangle D*EF*—C*EF* (equal D*EC*) is equal half the Parallelogram A*F*—B*F* (equal ABCD.) i. e. the Tri. D*EC* is equal half the Par. ABCD. Q. E. D.

THEOREM XVIII.

Containing the 35th, 36th, 37th, and 38th of Euclid.

Parallelograms, or Triangles, having the ſame or equal Baſes, and equal Altitudes, are equal.



The Parallelogram ABCD is equal to the Parallelogram AFGD; ſtanding on the ſame Baſe AD, and between the ſame Parallels AH, BG; i. e. they have the ſame or equal Altitudes.

Alſo, the Par. ABCD is equal to EFGH, having equal Baſes, AD=EH.

DEM. Now, AB=CD, and AF=DG. - - Th. 15.
And, the Angle BAF=CDG. - - - - - 6.
Wherefore, the Triangle ABF=DCG. - - - 8.
But, the Triangle ICF is common to both;
wherefore, the Trapezium ABCI=IFGD. - Ax. 7.
Add, to both, the Triangle AID; which is common;
the Parallelogram ABCD is equal to AFGD - - Ax. 6.

Again. The Par. ABCD, is proved equal to AFGD.
For the ſame reaſon, EFGH=AFGD, FG being common.
Therefore, the Par. ABCD=EFGH. - - - Ax. 3.
But, AD=FG and EH=FG (15); wh. AD=EH.-Ax.3.
Th. the Parallelograms, AC and FH, have equal Baſes.

Secondly. Draw the Diagonals BD, DF and EG.

Now, ſince every Triangle is equal to half a Parallelogram having the ſame Baſe and Altitude, - - - - Th. 17.
the Tri. ABD=half the Par. ABCD; &, AFD=half AFGD
Therefore, the Triangle ABD is equal to AFD. - - Ax. 4.

And, for the ſame reaſon, the Triangle ABD=EGH.
(For, the Parallelogram ABCD is equal to EFGH.)
And, their Baſes, AD, EH, are alſo equal. Q. E. D.

COR.

Cor. Equal Parallelograms or Triangles, having the ſame or equal Baſes, have the ſame Altitude; or, are contained between the ſame Parallels.

This Corollary is the converſe of the Theorem, and contains the 39th and 40th Propoſitions of Euclid.

N. B. From this Theorem, the 24th, 26th, and 27th Problems are deduced; which may be found particularly uſeful to Surveyors of Land, &c.; ſeeing, by its means, any Plot or Survey, though ever ſo irregular, in Figure, may be reduced, with the greateſt facility, and accuracy, to a Trapezium or Triangle; and, its Area obtained at one Operation. (See Prob. 24.)

Alſo, by its means, almoſt any right-lined Figure, as well as a Trapezium, may be divided into two equal Parts, by a Right Line from any detetmined Point, in any Side.

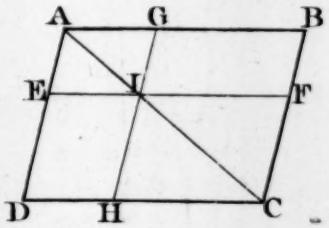
An Example, of which, will be found in the Appendix.

SCHOL. *Hence it is evident, that two Spaces may differ greatly in Circumference, yet contain the ſame Space; and alſo, that two Figures or Spaces, of equal Circuit, may contain very different Areas.*

THEOREM XIX. 43. Euclid.

In every Parallelogram, the Complements are equal.

In the Parallelogram ABCD; aſſume any point, as I, in the Diameter AC; through which, draw EF and GH, parallel to AB and AD.

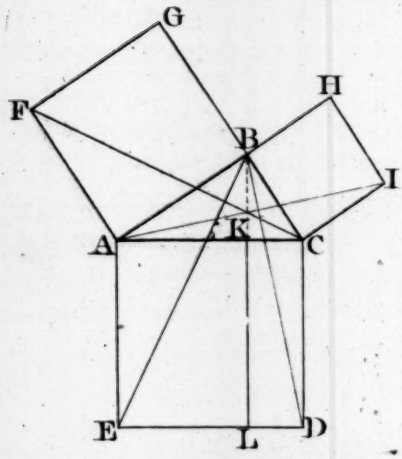


I ſay, the Complements, DI and IB, are equal to each other.

Dem. AC divides the Par. DB into two Tris. ABC=ACD. And the Parallelograms AGIE and IFCH, are divided into equal Triangles AGI=AIE, and IFC=ICH. - Th. 15. Now, if from the equal Triangles ABC, ACD, there be taken away the equal Triangles AGI, IFC=AIE, ICH, there will remain, the Par. DEIH equal to IGBF. - Ax. 7.

THEOREM XX. 47. Euclid.

In every Right-angled Triangle, the Square of the Hypothenuſe, is equal to the Squares, of the two Sides, containing the Right Angle.



Let ABC be a Right-angled Triangle.

On the Hypothenuſe AC, deſcribe the Square ACDE; and, on the Sides AB, BC, deſcribe the Squares AFGB and BHIC.

I ſay, the Square AD, of the Hyp. AC is equal to the two Squares, AG and BI, of the Sides, AB and BC.

From the Right Angle, B, draw BK perpendicular to AC; which, produce to L, parallel to CD. The Square AG is equal to the Rectangle AL; and, the Square BI is equal to the Rectangle KD. Draw FC & BE.

DEM. In the Triangles FCA, ABE, AF, AC are reſpectively equal to AB, AE; and they contain equal Angles. For, FAB=EAC (Ax. 9.) and, if BAC be added to both, the Angle FAC=BAE. - - - - - - - - - Ax. 6.
Wh. FC=BE, and the Triangle FCA=ABE. - Th. 8.

But, the Tri. FCA is equal to half the Square FB - 17. (for, they have the ſame Baſe, FA; and, they are between the ſame Parallels, GC and FA.)
And, the Tri. ABE (eq. FCA)=half the Rect. AKLE; (on the ſame Baſe, AE, & between the Parallels BL & AE)
Therefore, ſince the Tri. FCA=ABE (and they are each equal to half the Square AG, or the Rectangle AL) the Square AFGB is equal to the Rectangle AKLE - Ax. 5.

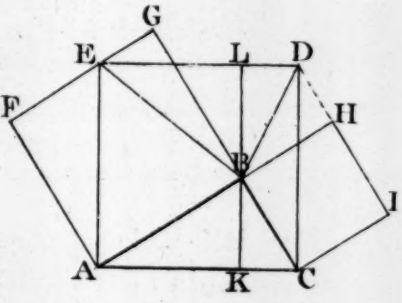
Again;

Again; the Square BI is equal to the Rectangle CKLD;
Draw AI and BD.

Then, in the Triangles AIC, CBD, the Sides IC, CA, are equal to BC, CD, reſpectively.
And, the Angle ICA=BCD (ACB being common)-Ax.6.
wherefore, the Triangle AIC=CBD. - - - Th. 8.
But, the Triangle IAC,= half the Square CH; - 17.
(having the ſame Baſe, CI, and between the Par^{s}.AH, CI)
And the Tri. CBD (eq. IAC)= half the Rect. KD - 17.
(being on the ſame Baſe, DC, & between the Par^{s}.LB, DC)
Wh. the Square BI is equal to the Rectangle KD-Ax. 5.
But, the Square AG was proved equal to the Rect. AL.
and AL+KD=AD (the Square of the Hyp. AC.)-Ax. 2.
Therefore, the Square AFGB+BHIC=ACDE. Q.E.D.

Or, it may be, as elegantly and more briefly, demonſtrated after the following manner.

Having deſcribed the Squares FB and BI, of the two Sides AB, BC, let the Square of AC, the Hypothenuſe, be inverted; as AEDC.



Through B, draw KL parallel to CD.
Join BD & BE; and produce IH to D.

Then, the Square AG is equal to the Rectangle AL; and the Square CH is equal to the Rectangle CL.

Dem. For, the Triangle AEB=half the Square AFGB - 17.
having the ſame Baſe, AB, and Altitude, BG.
The Tri. AEB is alſo equal half the Rect. AELK - ſame
having the ſame Baſe, AE, and Altitude, AK.
Th. the Square AG is equal to the Rect. AL. - - Ax. 5.

Again; the Tri. BDC=half the Square BHIC; - 17.
(on the ſame Baſe, BC, & between the Parallels DI, BC).
And the Tri. BDC is alſo equal half the Rect. KLDC;
(having the ſame Baſe, CD, and Altitude, LD.)
Wh. the Square BHIC = the Rect. KLDC. - Ax. 5.
But, the Rect. AL+LC= the Square AEDC. - Ax. 2.
Th. the Square AEDC (of the Hyp. AC) is equal to the Squares, AFGB and BHIC, of the two Sides, AB & BC.

COR. Hence; if, in a Triangle, the Square of one Side be equal to the Squares of the other two Sides, the Angle, contained by thoſe two, is a Right Angle.

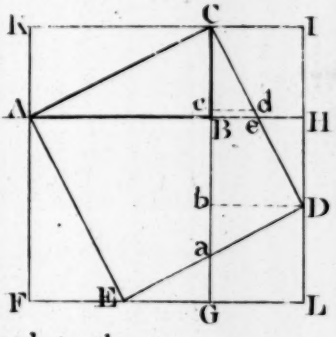
I have been more particular in the Demonſtration of this famous Theorem, than any of the preceding ones; not becauſe it is the laſt of this Book, but, becauſe it is ſo extraordinary in itſelf, and of ſuch ſingular uſe throughout the Mathematics. It is alſo of great uſe in the following Books of the Elements; inſomuch, that, without it, we ſhould not be able to advance much further. Pythagoras is ſaid to be the inventor of it; for joy of which, it is ſaid, that he offered a hundred Oxen to the Deity, who inſpired him with ſo noble, and ſo uſeful, a Theorem.

That there may not remain the leaſt doubt, in the mind of any, concerning the truth it contains, I have added another Figure; the Demonſtration of which is both ocular and mathematical.

EX. 3. Let the Square AEDC, of the Hypothenuſe AC, be inverted, as before; and, having drawn the Squares of the two Sides, AB, and BC; produce FA and FG, IC and IH, meeting in K and L; forming the Square FLIK.

The two Rectangles ABCK, and BGLH, are equal; being under equal Lines, AB=BG, and BC=BH.
And the four Triangles ACK, AFE, ELD, and DIC, are equal between themſelves, and alſo to the Rectangles, BL, BK; each being equal to half a Rectangle;
as ACK, equal half ABCK. - - - - - Th. 15.

But,



But, the large Square FLIK, is equal to the two Squares FB, and BI, added to the two Rectangles, BL and BK. - Ax. 2.

It is alſo equal to the Square AEDC, added to the four Triangles ACK, AFE, ELD, and DIC.

And, the four Triangles are equal to the two Rectangles, BL and BK.

Conſequently, the Square AEDC is equal to the two Squares, FB and BI, of the Sides, AB and BC.

Note. If the Squares, AFGB and BHIC, are cut according to this Figure, by inverting the Square AEDC; the Parts, ſo divided, will exactly cover the Square AEDC.

For, the Perpendicular Db being drawn, and bc made equal to bD (equal CI) draw cd parallel to bD.

Then, the Trapezium, AEaB, is common to both; the Tri. AFE is congruous with ABC, & EaG with Cdc; alſo, the Trap. CIHe will agree with Dbcd; and, CeB will coincide with Dab, which completes the Figure.

Therefore, the Square AFGB + BHIC = AEDC. - Ax. 2.

Having, in theſe twenty Theorems, gone through the firſt Book of the Elements of Geometry, (ſeventeen of which, contain the whole thirty-four, according to Euclid) it will not, I preſume, be impertinent, before we proceed further, to ſay ſomething of its utility. It is reaſonable for a perſon, who has gone ſo far, to aſk, of what uſe are all thoſe properties of Triangles and Parallelograms, &c.? which queſtion, though pertinent enough, it may not be ſo eaſy to give an anſwer to, which ſhall be ſatisfactory; unleſs we are acquainted with almoſt every branch of the Mathematics.

If the ſtudy of Geometry was of no other uſe, than for the enuring us to Demonſtration; by which we may have conviction in the queſt of Truth, it would be great. Geometry is Truth itſelf; the manner of reaſoning geometrically, is ſolid and convincing. It is neceſſary for a Perſon who would be a great Orator, at the Bar, &c. to ſtudy Geometry; it teaches how to range our Ideas, to ſtate the Premiſes fairly, and to draw Concluſions from them with clearneſs and certainty. But, when we conſider that it is an

an Introduction to the Mathematics, a Key to let in the firſt Ideas, to further and more extenſive knowledge, we ſhall purſue it with avidity.

The knowledge of the properties of Triangles and Parallelograms is very extenſive, and of great uſe in almoſt every mathematical ſcience; as Surveying, Navigation, Perſpective, Aſtronomy, and numerous others, depend on them. The Demonſtrations of the moſt ſimple general properties of Triangles, in Theo. 7, 8, 9, 11, 12, 13, and 14th, are neceſſary for the Demonſtration of others, as well as for reference, for the proof of what is aſſerted in other Works. The 10th, a general property, and the 20th, a particular one, are more ſublime properties of Triangles; and are, perhaps, the moſt perfect Demonſtrations that can be given; which are frequently quoted and referred to hereafter.

The affinity between Triangles and Parallelograms, in Theorems 15 and 17th, is the foundation of Menſuration of Superficies. The properties of Parallelograms and Triangles in the 18th, are very extenſive; the equality of the Complements of Parallelograms, in the 19th, is alſo of great utility; both, which, carry with them the ſtrongeſt conviction imaginable.

In reſpect of the properties of Right Lines, and Angles formed by their Interſections, in the firſt ſix Theorems, the knowledge of which, though in a manner ſelf-evident, is abſolutely neceſſary for attaining the properties of Figures. I ſhall juſt give one inſtance, as a ſpecimen, of what great things may be effected, from the knowledge inculcated by the fourth Theorem; viz. that the alternate Angles, formed by a Right Line cuting two parallel Lines, are equal.

It would hardly gain credit, with ſome, to aſſert, that it is poſſible from the meaſure of a ſmall portion of the Earth's Circumference to determine the whole, by means of that ſimple property; yet nothing is more certain, and that, without much more previous knowledge, than, that the ſame portion or ark of every Circle, gives or ſubtends equal Angles at the Center; and conſequently, equal Angles being formed at the Centers of Circles of any Radius, will cut off equal portions of the Circumference. (See the Theory of Plane Angles.)

It is very practicable to meaſure the diſtance between two places, on a level Plane; and, by means of an Inſtrument, contrived for that purpoſe, any ſpace or diſtance between two places, very remote, may be aſcertained, let the ground, between, be ever ſo irregular, interſperſed with Hills and Vallies; in which caſe, it muſt be obſerved, that the meaſure, ſo obtained, is not the real Diſtance between thoſe places, which can only be meaſured by a Right Line (ſee Def. 3, and 4.) and which is impoſſible to be had by an Inſtrument. And, let the Ground between two remote places be ever ſo apparently level, it is not a Plane, but a portion of a Sphere;

Sphere; and conſequently, the meaſure, by an Inſtrument, over the Sruface, is the meaſure of an Ark of a Circle, of which the Chord Line is the true Diſtance. (See Def. 22.)

This convexity of the Earth would be obvious, to ocular conviction, if its Surface was regular. In a perfect Calm, when the Surface of the Sea is at reſt, it is diſcernable, that a Ship at a diſtance, in going from Land, will gradually appear to deſcend; the Bottom or Hull of the Veſſel will ſoon be loſt to ſight; and, by means of Teleſcopes, the Sails and Rigging of the Ship may be ſeen, when the Hull is apparently ſunk in the Water; which alone, is ſufficient conviction, to any thinking Perſon, of the Earth's rotundity.

Now, although it is poſſible to meaſure a ſmall portion of the Earth's Circumference, with tolerable accuracy, it would be impoſſible to meaſure the whole; becauſe it is, in no part of it, entirely circumſcribed with Land; and there is no dependence on the meaſures taken at Sea; therefore it is impracticable, for ſeveral reaſons, to obtain its true Circumference by actual meaſurement.

It is eaſy to conceive, that two Right Lines or Poles erected perpendicular to the Earth's Surface, at any tolerable diſtance from each other, are not parallel between themſelves; becauſe, the Earth, being a Globe, or globular, every Right Line or Plane, which is perpendicular to its Surface, would, if produced, paſs through its Center. Conſequently any two Right Lines, which are perpendicular, would, if produced, form an Angle at the Center; which Angle, is greater or leſs in proportion to the diſtance of the Perpendiculars from each other (ſee Art. 1. Theory of Plane Angles.) Hence it is plain, from what has been ſaid, that nothing more is required, than to determine the Angle which two Perpendiculars would form at the Earth's Center, according to the Diſtance between them.

Suppoſe a ſmall portion of the Circumference (which may be repreſented by the Circle F G H) be meaſured, from B to D, where the Earth is as nearly on a level as poſſible. (By a ſmall portion, I would not be underſtood to mean leſs than two or three Degrees, or about 200 miles.)

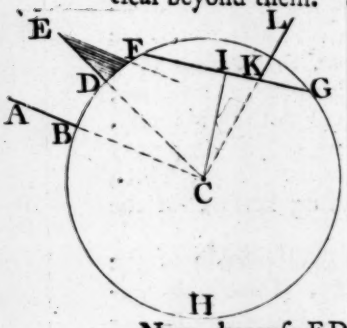
Let AB repreſent a Pole, or Stile, erected perpendicular to the Horizon, at B; it will conſequently, tend to the Center, at C. Now the Ark B D being meaſured, due North or South, and another Pole, D E, erected perpendicular to the Horizon, at D, it will alſo tend to the Center, C, making an Angle, ACE, with AB.

The whole myſtery is to determine that Angle.

From the immenſe diſtance and magnitude of the Sun, in reſpect of the Earth, its Rays, and conſequently the Shadows of Objects, on the Earth, are projected in Right Lines, parallel amongſt themſelves.

It

It muſt alſo be obſerved, that this operation can only be performed at or between the Tropics; becauſe, the Sun is never vertical beyond them. Suppoſe the Perpendicular AB to be erected.



When it is in the Meridian of that place, the Shadow of AB will be projected towards C, the Center of the Earth, for no Shadow of it can be projected on the Surface, the Sun being perpendicular over it. At the ſame Time, any other perpendicular Line, as ED, at a great diſtance from AB, will caſt a Shadow, as DF, in proportion to the length of the Perpendicular DE.

Now, becauſe ED is perpendicular to FD the Angle EDF is a Right one. (Def. 11.) Wherefore, the meaſure of ED being known, and the length of the Shadow DF taken, DF being conſidered as a Tangent, and DE the Radius, the Angle DEF is determined, by Trigonometry; or, by a Scale of equal parts, making a right-angled Triangle of the two given Lines, ED and DF (Prob. 13.) which will give the Angle DEF.

But, EF is parallel to AB; and EDC is a Right Line cuting them both; wherefore, the Angle ACE is equal to CEF (Th. 4.). Therefore, the Angle, at C is determined.

Now the Angle BCD being known, which, ſuppoſe equal two and a half or three Degrees; then, as ſo many Degrees is to 360, ſo is the Ark BD in Miles, &c. to the whole Circumference; by the Rule of Three, or Proportion.

Or, having deſcribed a Circle of any Radius, and made an Angle BCD, equal to DEF, which a Ray of Light makes with the Perpendicular DE (Prob. 4.) it will intercept a portion of the Circumference, BD, which being taken in your Compaſſes (beginning at B or D) ſee how oft it is contained in the Circumference FGH; which, multiplied into the number of Miles, meaſured on the Ark BD, will give the meaſure of the Earth's Circumference, in Miles.

Hence it is alſo evident, that, if a large portion of the Earth's Surface was a Plane, inſtead of being ſpherical (of which let FG be a Section) we ſhould, at any diſtance from I, where a Right Line, CI, from the Center, is perpendicular to FG, appear as if on a declining Plane; and the farther from the point I, the greater will be the inclination; inſomuch, that in going from I, towards F or G, we ſhould ſeem as if climbing an aſcent; making, at K, the acute Angle LKG, with the Plane FG. For, a Perſon, ſtanding upright at K, would tend towards the Center C, in the direction LC; and conſequently, would be more inclined to FG, the farther the point K is from the Perpendicular CI.

ELE-

ELEMENTS OF GEOMETRY.

BOOK II.

THE ſecond Book of Elements treats of the properties or powers of Right Lines, divided at pleaſure, and otherwiſe. It generally ſeems, at firſt, ſomewhat ſtrange and obſcure to beginners, who are unacquainted with Menſuration; but, if they carefully obſerve the inſtructions, I have given, concerning Rectangles, all difficulty will ſoon vaniſh. The method or manner of Demonſtration, is, in the firſt eight Theorems, both ocular and mathematical; it is alſo, in the ten firſt, numerical, and does not admit of the leaſt doubt to any, who are tolerably verſed in the fundamental rules of Arithmetic.

It is of eſpecial uſe in attaining a knowledge of the properties of the Circle, as contained in the third Book. For which reaſon, I adviſe the young Student, not to paſs it over ſlightly and with indifference, but to digeſt it carefully, and be ſure he underſtands it perfectly; the ten firſt particularly, before he advances further; elſe, he will frequently be obliged to turn back to them. Therefore, I adviſe him to get them by Heart, ſo, as to be clearly certain of the power of a Line, ſo and ſo divided, whenever it occurs in the courſe of this Treatiſe; it will greatly facilitate his purſuit of the remainder, and ſave him ſome trouble.

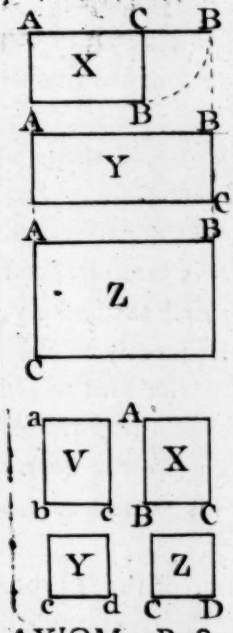
This ſecond Book treats alſo, more fully, of the Theory of Triangles, in general; ſhewing, as in the right-angled, the proportion which the Square of any Side of a Triangle, of any ſpecies, has to the Squares of the other two; by which

means, the Perpendicular, and, consequently, the Area of any Triangle may be obtained by calculation, in Numbers, by the measures of the Sides only, the Perpendiculars being otherwise unattainable. Likewise some other properties of Triangles and Parallelograms, which are not in Euclid.

I have made no alterations in this Book, nor abridgements; but have added some very elegant Theorems. The Demonstrations, in general, as well in the other Books as this, I have made as brief as is consistent with perspicuity, and the nature of Demonstration.

RECTANGLES and SQUARES are already defined. (See Def. 34, & 35, in the general Introduction to Geometry.)

A Rectangle, or right-angled Parallelogram, is said to be under, or contained under two Lines, which are the measures of its Sides.



If a Right Line be any how divided, as AB, in C, the Rectangle A C B or B C A is the same; and signifies a Rectangle under the two Segments, AC & CB; i. e. AC×CB; as X.

But the Rects. A B C and BAC, are different from that, and from each other, if the Line be divided unequally.

ABC □ signifies a Rectangle under the whole line, AB, and the Segment, B C, or AB×BC; as Y.

And, BAC means one under the whole line, AB, and the other Segment, A C, or BA×AC; as Z.

The middle letter is always twice meant.

AXIOM. Rectangles or Squares, contained under equal Right Lines, are equal.

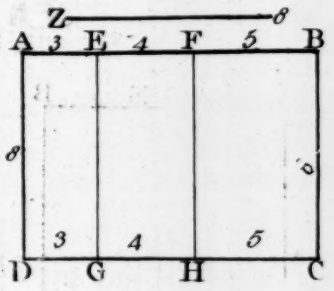
As V and X, under the Line a b, eq. AB, and b c, eq BC. And the Squares Y and Z, under c d, equal C D.

THEOREM I.

The Rectangle contained under any two Right Lines, is equal to all the Rectangles, under either Line, and the ſeveral Segments of the other, divided into any number of parts, at pleaſure.

Let AB and Z be the two given Lines; and let AB be divided, at pleaſure, in E and F.

I ſay, that $Z \times AE + Z \times EF + Z \times FB = AB$ multiplied into Z.



On the Line AB, conſtruct the Rectangle ABCD, whoſe Sides AD and BC are each equal to Z. - - - - - Pr. 18. Through E and F, draw EG and FH, parallel to AD and BC; dividing the Rect. ABCD into three Rect. AG, EH & FC.

Dem. Now, becauſe AD and BC are both equal to Z; Con
EG and FH are alſo equal Z. - - - Th. 15. 1. & Ax. 3.
For, ABCD is a Rectangle, by Conſtruction;
wherefore, AG, EH and FC are Rectangles.
But, the Rect. AC = all the Rects. AG, EH & FC - Ax. 2.
and the Sides AD, EG, FH and BC, are each equal Z.
Therefore, $Z \times AE + Z \times EF + Z \times FB = Z \times AB$.

In Numbers thus.

Let AB repreſent, by ſome Seale, 12 Feet, or any other meaſure; and, let Z be equal 8 of thoſe Parts.

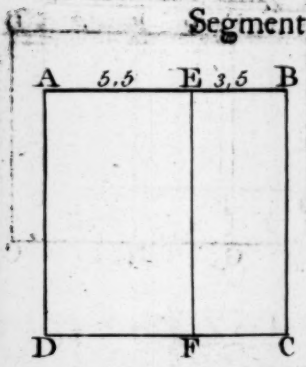
Let the Segment, AE be 3, EF 4, and FB 5.

Then, the Rect. AG, i. e. $Z \times AE$; or $8 \times 3 = 24$.
\+ the Rect. EH, i. e. $Z \times EF$, or $8 \times 4 = 32$.
\+ the Rect. FC, i. e. $Z \times FB$, or $8 \times 5 = 40$.
= the Rect. AC, i. e. $Z \times AB$, or $8 \times 12 = 96$.

Cor. If the two Lines are divided into Parts, at pleaſure; the Rectangle under the two whole Lines, is equal to all the Rectangles under the ſeveral ſegments of both Lines.

THEOREM II.

If a Right Line be divided, any how, into two parts; the Rectangles under the whole Line and each Segment, are equal to the Square of the whole Line.



Let AB be divided, at pleaſure, in E. I ſay, that AB×AE + AB×EB = AB×AB, i. e. the Square of AB.

On the given Line, AB, deſcribe the Square ABCD; and, through E, draw EF, parallel to AD.

Dem. Now, ABCD is a Square; by Conſtruction; wherefore, AD, and BC are, each, equal to AB - Def. 35.
And AEFD, EBCF, are Rectangles. (Con.)
wherefore EF is equal to AD, or BC; - - - - - - 15. 1.
But, the Square ABCD = the Rectangles AF, FB - Ax. 2.
Therefore, AB×AE + AB×EB = AB×AD, or AB □.

In Numbers, let the whole Line, AB, be 9;
and let it be divided in AE 5,5, and EB 3,5.

Now, the Rect. AF, i. e. AB×AE, or 9×5,5, = 49,5.
\+ the Rect. FB, i. e. AB×EB, or 9×3,5, = 31,5.
= the Square AC, i. e. AB×AB, or 9×9, = 81,0.

N. B. This Theorem is only a particular caſe of the former, viz. when the two given Lines are equal; and might have been a Corollary to the firſt; but I was not willing to break the order of Euclid unneceſſarily.

THEO-

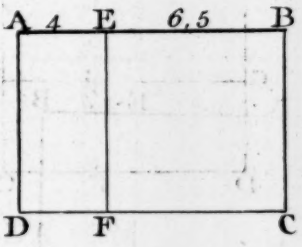
THEOREM III.

If a Right Line be divided at pleaſure; a Rectangle under the whole Line and either of the Segments, is equal to a Rectangle under the two Segments, added to the Square of the Segment, firſt taken.

Let AB be divided in E.

I ſay, that AB×BE=AE×EB+EB□.

Conſtruct the Rectangle ABCD, under the whole Line AB and the Segment EB; and draw EF parallel to AD.



Dem. The Rectangle AEFD added to the Square EBCF is equal to the Rectangle ABCD. - - - - Ax. 2.
But, the Rect. AF is under the Segments AE, EB, (eq. EF) and, FB is the Square of EB, the Segment firſt taken.
Th. AB×EB (or ABE□)=AE×EB (AEB□)+EB□
Alſo, AB×AE=AEB□+AE□, of the leſſer Segment.

Let the whole Line, AB be 10,5; let AE be 4, & EB, 6,5.

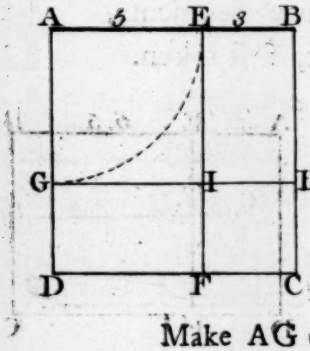
Then, the Rect. AF, i. e. AE×EB, or 4×6,5=26
\+ the Square FB, i. e. EB×EB, or 6,5×6,5=42,25
= the Rect. ABCD, i. e. AB×EB, or 10,5×6,5=68,25

N. B. I have given this Line, in Numbers, fractional; and alſo divided the former, fractional-wiſe, in order to ſhew, that it will hold true in Numbers (as in Lines) any how divided; which, I adviſe the young practitioner to try, by various fractional diviſions, if he be converſant in Decimals; it will be of ſome ſervice, and make him more perfect in the whole Theory of this ſecond Book.

THEO-

THEOREM IV.

If a Right Line be divided at pleaſure; the Squares of the two Segments, added to two Rectangles under the Segments, are equal to the Square of the whole Line.



Let A B be the given Line; divided, at pleaſure, in E.

Then will the two Squares of AE & EB added to two Rectangles under AE & EB, be equal to the Square of AB.

On AB, conſtruct the Square ABCD. and draw EF parallel to AD.

Make AG equal AE, and draw GH parallel to AB.

DEM. Then, becauſe ABCD is a Square (Con.) the Sides AD, DC, & BC, are each equal, to AB; and the Angles A, B, C, and D are all Right ones. - - - Def. 35.
Wh. ſince AG was made equal to AE, GD is equal to EB.
But, GH is parallel to AB, and EF is parallel to AD; wh. HC, and FI, IH, and FC, are each equal to EB.
EI and GI are, for the ſame reaſon, each equal to AE.
And, the Angles AEI, IHC; AGI, and IFC, are each equal to ABC, and ADC. - - - - 2. Th. 4. 1.
Wherefore, AEIG and IHCF are Squares of AE and EB, and GF, EH are Rectangles under the Segments AE & EB
And, they are all equal to ABCD, the Square of AB. Ax. 2
Therefore $AE\square + EB\square + 2\ AE \times EB = AB\square$.

Let the Line AB be 8, divided at E, in 5 and 3.

Then, the Square GE,	i. e. AE □,	or	$5\times5, = 25$
+ the Square FH,	i. e. EB □,	or	$3\times3, = 9$
+ two Rect. GF & EH.	2 AEB□,		$2.5\times3, = 30$
= the Square ABCD,	i. e. A B □,	or	$8\times8, = 64$

Cor. If a Line be equally divided, the Rectangle under the Segments is a Square, and is equal to the Square of either Segment.

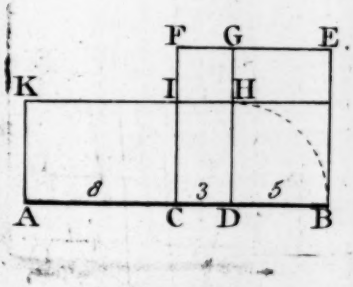
Hence, the Square of a whole Line is equal to four times the Square of half the Line. $4\times4\times4=8\times8,=64$.

THEOREM V.

If a Right Line be biſected, and alſo cut unequally at pleaſure; the Rectangle, under the two unequal parts added to the Square of the intermediate part, i. e. the difference between the equal and unequal, is equal to the Square of half the Line.

Let AB be cut equally in C, and unequally in D.

Then, the Rectangle under AD and DB, added to the Square of CD, is equal to the Square of AC or CB.



On CB, half AB, deſcribe the □CBEF. Draw DG parallel to CF; make DH equal DB, and draw KH parallel to AB, and AK parallel to CF.

Dem. Now CFEB is a Square (Con.) and AC=CB - Hyp. wh. BE=AC (Ax. 3.) & AK, CI, are each =DH-Def. 7. But DH=DB (Con.) wh. AK & CI are each =DB-Ax. 3. Th. the Rect. AKIC is equal to the Rect. DGEB. - Ax. Add, to both, the Rect. FD; &, AI+FD=FD+GB - 6. but AI+ID, eq. AH, =ADB□, & FGHI is the □ of CD; for, DG is parallel to CF, and are each equal to CB - Con. alſo, KH is parallel to AB, and CI is equal to DB; conſequently, IF and GH are each equal to CD. - Ax. 7. Th. the Rect. AH + the Square FH=the Square CFEB. - 2. i. e. AD×DB+CD□=AC, or CB □.

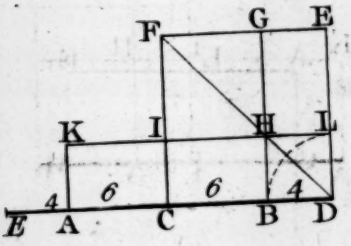
Let

Let the whole Line AB be 16; AC=CB each =8; let CD be 3, and DB 5; then AC+CD, =AD, =11.

Then, the Rect. AH, i. e. ADB □, or 11×5=55
+ the Square FH, i. e. CD □, or 3×3= 9
= the Square CFEB, i. e. CB □, or 8×8=64

THEOREM VI.

If a Right Line be divided into two equal Parts, and then produced at pleaſure, or another Line be added; the Rectangle contained under the whole compounded Line and the Part added, together with the Square of half the given Line, is equal to the Square of the half Line and the Part added, in one Line.



Let the given Line AB be biſected in C, and let BD be added to it, at pleaſure.

I ſay, the Rectangle ADB added to CB Square is equal to the Square of CD.

On CD, deſcribe the Square CDEF; and on AD the Rectangle AKLD; by drawing AK parallel to CF, and KL to AD, making DL equal to BD. Draw BG parallel to DE, and join FD.

Dem. The Rect. AI=IB (Ax.) and IB=GL (19. 1.) therefore, AI is equal to GL. - - - - - - Ax. 3.
Conſ. the Rect. AI+IB+BL=IB+BL+GL - Ax. 6.
And, if to both, be added the Square IFGH, of CB.
The Rect. AKLD+IFGH=CFED. - - - Ax. 2.
i. e. ADB□, or AD×DB+CB□=CD□.

Or

Or it may be demonſtrated as the foregoing.

Produce BA, and make A*E* equal to BD.

Then is *E*D biſected, in C, and unequally cut, in B.

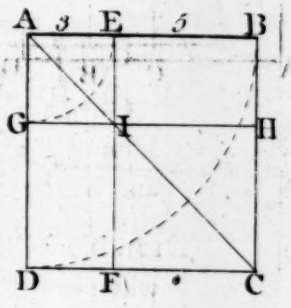
Wherefore, the Rect. *E*BD, under the unequal parts, (equal ADB) i. e. *E*B×BD, + CB□ = CD□.

Let AB be 12, biſected in C, AC, 6, and CB, 6, and let BD, the part added, be 4.

Then, the Rect. AKLD, i. e. AD×BD, or 16× 4= 64
\+ the Square IFGH, i. e. CB□, or 6× 6= 36
= the Square CFED, i. e. CD□, or 10×10=100

THEOREM VII.

If a Line be divided, equally or unequally, at pleaſure; the Square of the whole Line, added to the Square of either Segment, is equal to two Rectangles, under the whole Line and that Segment, together with the Square of the other Segment.



Let AB be divided, any how, in the point E.

Then, the Square of AB, added to the Square of AE, is equal to two Rectangles under AB and that Segment, added to the Square of EB, the other Segment.

Deſcribe the Square ABCD; through E, draw EF parallel to AD; make AG equal AE, and draw GH, parallel to AB; and join AC.

Dem. The Rect. DI = IB (19. 1.) add GE to both; & DE = GB
But, the Rectangle DE + GB = DI + IB + 2GE; - Th. 3.
and, if FH, i. e. the Square of EB, be added;
they are equal to the Square ADCB + GE. - - Ax. 2.
But, the Rectangles DE, GB, are under the whole Line, AB, and the Segment AE; for, AD = AB and AG = AE-Con.
Therefore 2BA×AE + EB□ = AB□ + AE.

Alſo, DH + FB + GE = DB + FH;
or 2ABE□ + AE□ = AB□ + EB□.

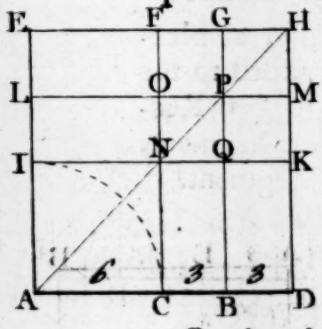
Let the whole Line, AB, be 8; let AE be 3, and EB 5.

Then the Square ABCD, of AB, i. e. $8\times8=64$
+ the Square AEIG, of AE, i. e. $3\times3=9$ } $=73$

But, the two Rects. DE, GB, i. e. $2\,AB\times AE$, $8\times3=48$
+ the Square IFCH, of EB, ——— $5\times5=25$
is also $=73$

THEOREM VIII.

If a Right Line be divided, any how, in two Parts; four Rectangles under the whole Line and either Segment, added to the Square of the other Segment, is equal to the Square of a Line, compounded of the whole Line and the Segment first taken.



Let AB be divided, at pleasure, in C; and, if CB be the Segment taken, make BD equal to BC.

Then, four Rectangles under AB and CB added to the Square of the Segment AC, will be equal to the Square of AD.

On the whole Line, AD, construct the Square AEHD. Draw CF and BG, parallel to AE; make AI equal AC, and AL equal to AB, and draw IK, LM, parallel to AD.

DEM. Now CF & BG are parallel to AE; & IK, LM, to AD; wherefore, FG, GH, HM, & MK, &c. are each equal CB; consequently, FP, GM, OQ, and PK are equal Squares. And, EO, LN, NB, and QD are equal Rectangles.

But the Rect. $EO+FP=ABC\square$, or $AB\times BC$.

Cons. the four Rectangles EO, LN, &c. added to the four Squares FP, GM, &c. are equal to four times $AB\times BC$; and, if the Square IC (of AC) be added, they are all equal to the Square AEHD, of the Line, AD, compounded of AB and the Segment CB, equal BD. - Ax. 2.

i. e. $4\,ABC\square$ or 4 times $AB\times BC+AC\square=AD\square$.

Let AB be 9, divided at C, in 6 and 3.
and if BD be equal CB, then AD will be equal 12.

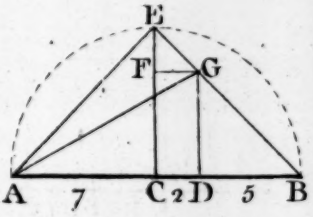
Then, the Rect. ABC, or AB×BC, i. e. 9×3=27.
Conſ. 4 Rectangles, AB×BC is equal to 108
+ the Square of AC, —— 6×6 = 36
= the Square of AD, —— 12×12=144

THEOREM IX.

If a Right Line be divided, into two equal and two unequal Parts; the two Squares of the unequal Parts are, together, double the Square of half the Line, together with the Square of the intermediate Part.

Let AB be biſected in C, and cut unequally in D.

Then, the Square of AD, added to the Square of DB; is equal to twice the Square of AC, added to twice the Square of CD, the difference between AC and AD.



Draw CE perpendicular to AB, and equal to AC or CB.
Join AE and EB, and draw DG parallel to CE.
Through G, where DG cuts EB, draw FG parallel to AB; and, laſtly, draw AG.

Dem. Now ſince ACE is a right Angle, & CE is equal AC, the Angles CAE and AEC are half right. - C. 3. 10. 1.
And AE□=AC□+CE□; i. e. equal 2AC□. - 20. 1.
for, ACE is a Right Angle, and AC=CE. - - Con.
And, becauſe CG is a Rectangle, FG=CD;
and FE is alſo equal CD; for EFG is a Right angle;
FEG (eq. FEA) is half right; th. the Angle FEG=EGF
conſequently, FE is equal to FG (equal CD) C. 3. 9. 1.
and EG□=EF□+FG□,=2 CD□. - - - 20. 1.

Now, AE□ = 2 AC□ ; and EG□ = 2 CD□ - above
But AEG is a right Angle, and AG□ = AE□ + EG□;
i. e. equal 2 AC□ + 2 CD□.
Alſo, AG□ = AD□ + DG□; for ADG is a right Angle.
And DG = DB ; for, GDB is a right Angle;
and DBG = DGB, equal CEB, half a Right one. - 4. 1.
Th. AG□, eq. AE□ + EG□, i. e. eq. 2 AC□ + 2 CD□;
is alſo equal to AD□ + DB, equal DG, □.
Therefore, AD□ + DB□ = 2 AC□ + 2 CD□. Q. E. D.

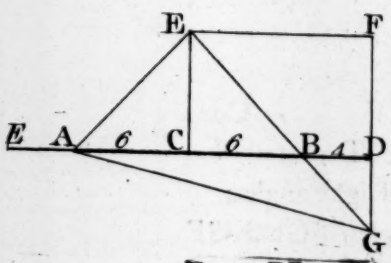
Let the whole Line AB be 14, divided equally, in C, and cut unequally, in D, in 9 & 5; AC 7, CD 2, & DB 5.

Then, AD□, i. e. 9×9=81
+ DB□ 5×5=25 } = 106
And AC□, i. e. 7×7=49
+ CD□ 2×2= 4 } = 53, ×2 = 106.

THEOREM X.

If a Right Line be biſected, and then produced at pleaſure; the Square of the whole compounded Line, together with the Square of the additional Part, will be double the Square of the half Line and the Part added, together with twice the Square of the half Line.

Let AB be biſected in C, and let BD be added.



I ſay, that the Square of AD added to the Square of BD, is equal to twice the Square of CD, added to twice AC Square.

Make ACE a Right angle, & CE eq. AC. Compleat the Rectangle CEFD, and produce the Side FD.

Draw EB, cuting FD in G; and laſtly, draw AE & AG.

DEM.

DEM. Then, becauſe ACE is a R. Angle, (con.) & AC=CE the Angles CAE and AEC are half Right. - C. 3. 10. 1.
And, for the ſame reaſon, CEB and EBC are half right.
And, becauſe CEFD is a Rectangle, EF=CD;
and ſince EFG is a Right Angle, and FEG (eq. DBG, eq. EBC) is half right, FGE is half right. - - Ax. 3.
wherefore, FG=FE, eq. CD; and DG=DB - C. 3. 9. 1.
Now, ſince ACE is a right Angle, and CE=AC
AE□=AC□+CE□; i. e. equal 2 AC Square. - 20. 1.
And, EG□,=EF□+FG□; i. e. equal 2 CD□.
But, AEG is a R. Angle; for AEC, CEG are half right.
wh. AG□,=AE□+EG□; i. e. equal 2 AC□+2 CD□
But, ADG is a R. Angle; conſ. AG□=AD□+DG;
and, ſince BD=DG, BD□=DG□; - - - Ax.
Therefore, AD□+BD□=2 AC□+2 CD□.

This Propoſition may alſo be demonſtrated as the former.

Let the given Line AB be biſected in C and let BD be added; alſo make *AE* equal BD in BA produced.

Then, becauſe *E*D, is biſected in C and cut unequally in B,
*E*B□+BD□=2 CD□+2 CB□.
But *E*B=AD; for A*E* was made equal to BD.
Wh. *E*B□+BD□ (eq. AD+BD□)=2 CD□+2 CB□.

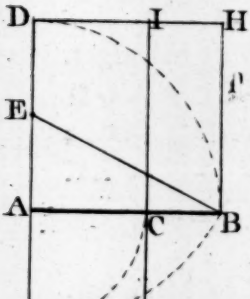
Let AB be 12; divided, in C, equally; and let BD be 4.

Then,	the Square of AD, i. e.	16×16=256	=272
+	the Square of BD,	4× 4= 16	
But,	the Square of CD,	10×10=100	
+	the Square of AC,	6× 6= 36	
		136×2=272	

THEO-

THEOREM XI.

To divide a Right Line, in ſuch wiſe, that the Rectangle under the whole Line and one Segment, ſhall be equal to the Square of the other; conſequently, the greater Segment.



AB is the Line given, to be divided.

It is required ſo to cut AB, in C, that the Rectangle, ABC, under the whole Line AB and the Segment BC, ſhall be equal to the Square of AC.

At the extreme A, draw AD perpendicular to AB.

Make AD equal AB, and biſect it in E; join EB.

Produce DA. Make EF equal EB, and AC equal to AF.

Then is AB ſo divided, in C, that $AB \times BC = AC\square$.

Compleat the Squares ADHB and AFGC, and produce GC to I.

DEM. Then, becauſe AD is biſected in E, and AF is added, the Rect. DFA, i. e. $DF \times AF + AE\square = EF\square$. - Th. 6.

But, $EF = EB$; wh. $EF\square = EB\square$; i. e. eq. $AB\square + AE\square$.

Conſ. $DF \times AF + AE\square = AB\square + AE\square$.

Take $AE\square$ from both, and there remains $DF \times AF = AB\square$.

But, the Rect. DIGF. is under the whole Line DF, and the Segment AF; and ADHB is the Square of AB; wherefore, the Rect. FI = the Square AH; and if the Rect. ADIC, which is common to both, be taken away, there is left the Rect. CIHB equal to the Square ACGF.

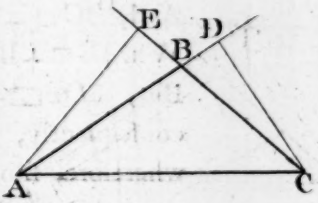
i. e. the Rect. ABC, or $AB \times BC = AC\square$.

This Propoſition cannot be exemplified in Numbers; i. e. the point C. cannot be found, nor the proof aſcertained arithmetically; for there is no Number, whatever, can be ſo divided, that the product of the whole Line multiplied by one part, ſhall be equal to the Square of the other.

Therefore, a Line ſo divided is incommenſurable.

THEOREM XII.

In an obtuſe angled Triangle, the ſquare of the Side which ſubtends the obtuſe angle, exceeds the ſquares of the other two Sides, by two Rectangles, under either Side containing the obtuſe Angle, and the part intercepted between the obtuſe Angle, and a Perpendicular, let fall from the adjacent Angle, to the ſame Side, produced.



ABC is an obtuſe angled Triangle. Produce the Sides AB and CB; and to them, draw the Perpendiculars AE and CD.

I ſay, the Square of AC exceeds the Squares of AB and BC, by two Rectangles ABD, or CBE.

DEM. Firſt, AC□ is equal to AD□+DC□. - - - 20. 1.
and AD□=AB□+BD□+2 ABD▭ - - - - - 4. 2.
wherefore, AC□=AB□+BD□+CD□+2 ABD▭.
But BC□ is equal to CD□+BD□.
Therefore, AC□=AB□+BC□+2ABD▭.

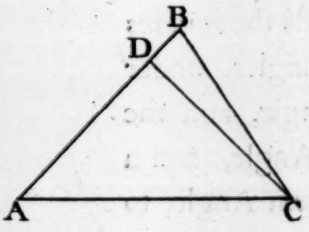
In the ſame manner, AC□ may be proved equal to AB□+BC□+2CBE Rectangles.

Conſequently, the Rectangles ABD and CBE are equal.

THEO-

THEOREM XIII.

In every Triangle, the Square of the Side ſubtending an acute Angle, is exceeded by the Squares of the other two Sides, which contain the acute Angle, by two Rectangles under either of thoſe Sides, and a part of it, intercepted between the acute Angle and a Perpendicular, let fall from the oppoſite Angle.



In the Triangle ABC, draw a Perpendicular, CD, to the Side AB.

Then, the Square of AC, is exceeded by the two Squares of AB and BC, by two Rectangles, ABD, under the whole Side, AB, and the Segment BD.

And, $AB\square + AC\square = BC\square + 2\,BA \times AD$.

DEM. Firſt, $AB\square = AD\square + DB\square + 2ADB\square$. - Th. 4.

And, $BC\square = CD\square + DB\square$. - - - - - - - 20. 1.

Wh. $AB\square + BC\square = AD\square + DC\square + 2DB\square + 2ADB\square$

But, $ADB\square + DB\square = ABD\square$. - - - - Th. 3.

conſequently, $2\,ADB\square + 2\,DB\square = 2\,ABD\square$.

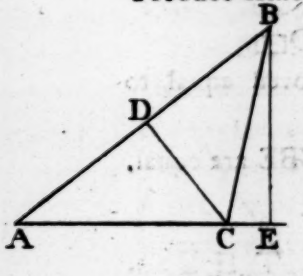
wherefore, ſubſtituting theſe for the other;

then, $AB\square + BC\square = AD\square + DC\square + 2\,ABD\square$.

But, $AC\square = AD\square + DC\square$. - - - - - - 20. 1.

Therefore $AB\square + BC\square = AC\square + 2\,ABD\square$.

Secondly, thus, more briefly, when an Angle is obtuſe. Produce either Side, AC, and from B draw BE perpendicular to it.



Now, $AB\square = BC\square + AC\square + 2ACE\square$

Add, on both ſides, the Square of AC.

Then $AB\square + AC\square = BC + 2\,AC\square + 2\,ACE\square$. - - - - - - - Ax. 6.

But, $2\,ACE\square + 2AC\square = 2EAC\square$;

or $2\,AE \times AC$. - - - - - Th. 3.

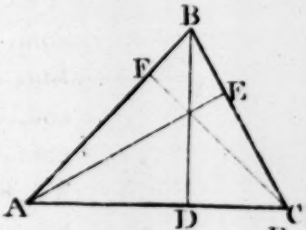
Therefore, $AB\square + AC\square = BC\square + 2\,AE \times AC$.

From which it is obvious, that the Rect. $BAD = EAC$.

By the first method it may be demonstrated; that the square of BC, is exceeded by the squares of AB and AC; by two Rectangles BAD; and by the second, the squares of AB and AC, exceeds the square of BC, by two Rectangles EAC; therefore the Rectangle BAD is equal to EAC.

SCHOL. *From these two Theorems it is evident, that if the square of one Side of a Triangle exceeds the squares of the other two, the Angle subtended by that Side, and contained by the other two, is obtuse; and, if the square of one Side, be exceeded by the squares of the other two; it subtends an acute Angle; consequently, when the square of one Side is equal to the squares of the other two, the Angle it subtends is a Right one.*

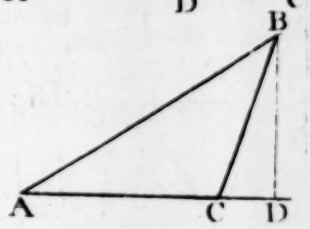
Cor. Hence, the Perpendicular, and consequently the Area of any Triangle, may be found or obtained, by the measure of the Sides only.



In the Tri. ABC, $AC\square + BC\square = AB\square + 2AC \times CD$. Wherefore, $AC\square + BC\square - AB\square = 2AC \times CD$; by Th. i, e. if $AB\square$ be subtracted from the sum of $AC\square + BC\square$, the remainder will be two Rectangles, under AC & DC. Wherefore, if half that Product be divided by the Side AC, the Quotient will be the Segment DC.

But $BC\square = BD\square + DC\square$. Or, $AB\square = AD\square + BD\square$. wherefore, $BC\square - DC\square = BD\square$. - - - - - 20. 1. Cons. having subtracted the square of DC from $BC\square$, or of AD from $AB\square$, the remainder will be the square of the Perpendicular BD, the square Root of which, gives the Perpendicular required.

By the same means, either of the other Perpendiculars, AE or CF, may be found.



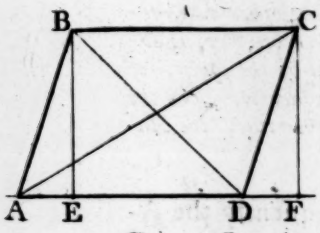
2ndly. When the Perpendicular falls without the Triangle, from the acute Angles of an obtuse angled Triangle, it is thus found.

In the obtuse-angled Triangle ABC, let fall the Perpendicular BD to the Side AC, produced.

Then, $AB\square = AC\square + CB\square + 2AC \times CD$. - - - - Th. 12. Wherefore $AB\square - AC\square + CB\square = 2AC \times CD$; half of which Sum being divided by the Side AC gives CD. But, $CB\square = BD\square + CD$ (20. 1.) wh. $CB\square - CD\square = BD\square$. the square Root of which gives the Perpendicular BD.

THEOREM XIV.

In every Parallelogram, the ſum of the Squares of the two Diagonals is equal to the Squares of all the Sides, together.



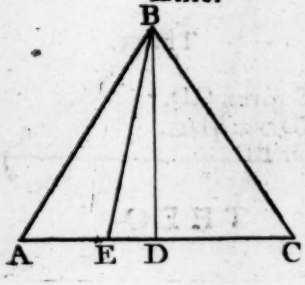
Let ABCD be a Parallelogram.

Draw the Diagonals, AC and BD; produce the Baſe AD, and draw the Perpendiculars BE and CF.

DEM. In the obtuſe angled Triangle ACD;
AC□=AD□+DC□+2ADF□. - - Th. 12.
And, in the Tri. ABD; BD□=AB□+AD□−2DAE□
But, EBCF is a Parallelogram, wh. EF=BC. - 15. 1.
and, AD=BC; wherefore AD=EF - - - Ax. 3.
conſ. AE=DF, ED being common; wh. ADF=DAE□
Therefore, as much as AC□ exceeds AD□+DC□, viz. by the Rect. ADF, twice; by ſo much is BD□ exceeded by AB□+AD□, eq. BC□; viz. by 2DAE□=2ADF.
Th. AC□+BD□=AB□+BC□+AD□+DC□.

THEOREM XV.

In Iſoſceles Triangles, the Square of one of the equal Sides is equal to the Square of any Line, drawn from the Vertex to the Baſe, added to a Rectangle under the Segments of the Baſe made by that Line.



In the Iſoſceles Triangle ABC, if BD be perpendicular, it biſects the Baſe, and is evident; for AB□=BD□+AD□; i. e. ADC□, or AD×DC. - - 20. 1.

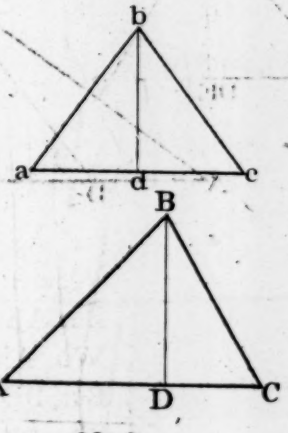
Let BE be drawn, at pleaſure.

I ſay, the Square of AB is equal to BE Square, added to the Rectangle AEC.

DEM. AB□ is equal to BD□ + AD□. - - - - 20. 1.
and AD□ is equal to AEC▭ + DE□. - - - - 5. 2.
(for AC, is cut equally in D, and unequally in E).
wherefore, AB□ = BD□ + DE□ + AEC▭.
But, BE□ = BD□ + DE□. - - - - - - - - 20. 1.
Therefore, AB□ = BE□ + AE×EB, or AEC▭.

THEOREM XVI.

If a Perpendicular be drawn from any Angle of a Triangle to the oppoſite Side; the Squares of the Sides, containing that Angle, added to the Squares of the alternate Segments of the Baſe, are equal; and the difference between the Squares of the Sides, is equal to the difference of the Squares of the Segments.



In Iſoſceles Triangles the thing is manifeſt.

In the Scalene Triangle ABC, draw the Perpendicular BD.

Then, the Square of AB added to DC Square is equal to BC Square added to AD Square.

And the difference, between the Squares of AB and BC, is equal to the difference, between the Squares of AD and DC.

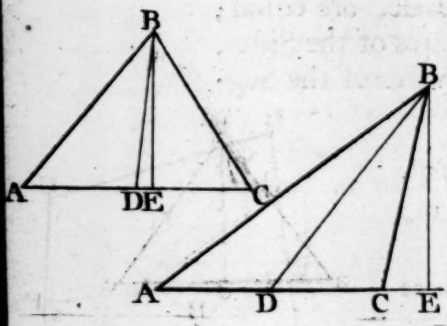
DEM. For, AB□ = AD□ + BD□.
And, BC□ = BD□ added to DC Square. - - 20. 1.
for the Angles, ADB and BDC, are Right ones. - Con.
Wh. AB□ + BD□ + DC□ = BC□ + BD□ + AD□.
that is, AB□ + BC□, is equal to BC□ + AB□. - Ax. 3.
Let there be taken, from both, BD□, which is common; there is left AB□ + DC□ equal to BC□ + AD□.

2nd. AD□ = AB□ − BD□; & DC□ = BC□ − BD□.
Therefore, AD□ − DC□ = AB□ − BC□.

THEOREM XVII.

If any Side of a Triangle is biſected, and a Right Line be drawn from the oppoſite Angle to the biſecting Point; the Squares of the other two Sides of the Triangle, will be equal to twice the Square of the biſecting Line, added to half the Square of the Side biſected.

If the Triangle be Iſoſceles, the thing is clear; for the Squares of the equal Sides, are equal to twice the Square of the Perpendicular, added to twice the Square of half the Baſe.



Let the Triangle ABC be Scalene; let the Side AC be biſected in D; and let BD be drawn.

Then the Squares of AB and BC, together, are equal to twice BD Square, added to twice AD or DC Square.

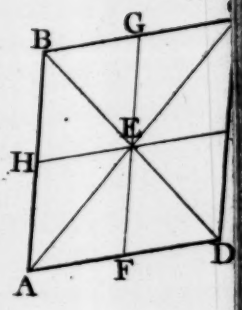
Draw the Perpendicular BE.

DEM. $AB\square = AE\square + BE\square$; and $BC\square = BE\square + EC\square$.
Wherefore, $AB\square + BC\square = AE\square + EC\square + 2\,BE\square$.
But $AE\square + EC\square = 2\,AD\square + 2\,DE\square$. - - 9. & 10.
Wh. $AB\square + BC\square = 2\,AD\square + 2\,DE\square + 2\,BE\square$.
But, $BD\square = BE + DE\square$; conſ. $2BD\square = 2BE\square + 2DE\square$
Th. $AB\square + BC\square = 2\,BD\square + 2AD\square$, equal $ACD\square$, or half $AC\square$. Q. E. D.

By this Theorem, may alſo be demonſtrated the 13th; viz. The Squares of the two Diagonals of every Parallelogram, are equal to the Squares of all the four Sides.

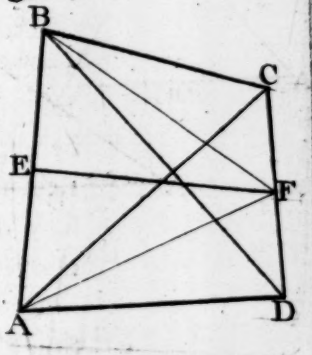
In the Par. ABCD (having drawn the Diagonals) through the Center, E, draw FG and HI, parallel to AB and AD.

In the Tri. AED, $AE\square + ED = 2EF\square + AF\square + FD\square$
conſ. $AE\square + ED\square = AH\square + AF\square + FD\square + DI\square$.
And, $BE\square + EC\square = HB\square + BG\square + GO\square + CI\square$.
But, $AE\square + EC\square = \frac{1}{2}AC\square$; &, $BE + ED\square = \frac{1}{2}BD\square$.
Alſo, $AF\square + FD\square = \frac{1}{2}AD\square$; & $AH + HB\square = \frac{1}{2}AB\square$.
and, the □s of BG, GC, CI, & $ID = \frac{1}{2}BC + \frac{1}{2}CD\square$.
Th. $AC\square + BD\square = AD\square + AB\square + BC\square + CD\square$.



THEOREM XVIII.

In any Trapezium, if the middle Points of two oppoſite Sides be joined by a Right Line; the ſum of the Squares of the two other Sides, together with the Squares of the Diagonals, is equal to the ſum of the Squares of the biſected Sides, added to four times the Square of the Line joining the middle Points.



Let the Sides AB and CD of the Trapez. ABCD be biſected in E & F; and draw EF.

I ſay, the Squares of the two Sides, AB and CD, added to four times the Square of EF, is equal to the Squares of BC and AD, together with the Squares of AC and BD.

Draw AF and BF.

Dem. Now, $AF\square + BF\square = 2AE\square + 2EF\square$. - Th. 17.
and, $AD\square$ added to $AC\square = 2AF\square$ added to $2FD\square$;
alſo, $BC\square$ added to $BD\square = 2BF\square$ added to $2FD\square$.
conſ. $AD\square$ added to $AC\square$ added to $BC\square$ added to $BD\square$ is equal to $2AF\square$ added to $2BF\square$ added to $4FD\square$.
But, $AF\square$ added to $BF\square = 2AE\square$ added to $2EF\square$.
conſ. $2AF\square$ added to $2BF = 4AE\square$ added to 4 EF Square.
But, 4 AE Square is equal to AB Square;
and, 4 FD Square is equal to CD Square. - Cor. to 4.
Therefore, $AD\square + AC\square + BC\square + BD\square = AB\square + CD\square$ + 4 FF Square. Q. E. D.

ELEMENTS OF GEOMETRY.

BOOK III.

THE knowledge of the properties of a Circle is the ſubject of the third Book of Elements. As a Circle is the moſt perfect of Plane Figures, the properties peculiar to it, are moſt extraordinary: ſome of which are really ſurprizing, and are very extenſive in their application.

The general Definitions of a Circle, and its attributes, are ſo well known, and, as they are given in the General Introduction, in Def. 19, and the four following, it would be impertinent to repeat them here; ſave only what are particularly neceſſary. The moſt ſelf-evident Propoſitions I have reduced into Axioms, for the more expeditiouſly attaining the knowledge of other, more eſſential, Properties.

My chief aim, throughout the whole, being to render the Study of Geometry eaſy, and agreeable to young Students; for which end, I conſider, brevity, if clear, moſt conducive thereto; and am perſuaded, that if the knowledge of the moſt ſimple properties of Figures can be attained, by any other means, Demonſtration is uſeleſs.

If a perfect knowledge of all that is contained in the Axioms of this third Book, be not acquired, by a bare recital, and inſpection of the Figure, I ſhall pronounce that perſon's capacity inſufficient, to purſue the Study of the more ſublime properties of the Circle; and, if they are clearly evident without, of what uſe is Demonſtration?

Where

Where the truth or evidence of an Aſſertion cannot be perceived, otherwiſe, Demonſtration is abſolutely neceſſary; but, where the evidence of ſight is ſufficient, the other is, in a great meaſure, uſeleſs.

There are ſome critical Geometricians who will not admit of any thing without Demonſtration; although they cannot but acknowledge many Propoſitions to be ſelf-evident truths. It would, perhaps, be no eaſy matter to give geometrical Demonſtration, that all Right Lines drawn from the Center of a Circle to the Circumference are equal; at the ſame time, no Perſon of common ſenſe can deny it, who knows any thing of the Figure, and geneſis of a Circle. If, therefore ſomething muſt be given or granted, why not others, which are as clear and ſelf-evident? for my part, I freely own, I can ſee no reaſon againſt it; and have, therefore, purſued the readieſt method for attaining the end aimed at; viz. to acquire a knowledge of Geometry in the moſt eaſy and familiar manner poſſible, by diveſting it of all that is ſuperfluous and unneceſſary; I mean unneceſſary Demonſtration, of what is clear without it.

I think it, however, neceſſary to apologize for the liberty I have taken in abridging it. Inſtead of 31 Theorems, in this third Book, according to Euclid, I have no more than 15, from Euclid; the reſt are diſpoſed of in the manner following.

The 2nd, the 5th and 6th, the 10th and 13th, are made Axioms.

The 9th is a Corollary to the 5th.

The 18th and 19th are Corollaries to the 8th; and the 23d and 24th may be deduced from the 3rd.

The 11th and 12th are both included in the 7th.

The 26 and 27th are the 2nd Corollary to the 9th; the 28th and 29th, Corollary 2nd of the 3rd; and the 37th is the 3rd of the 16th.

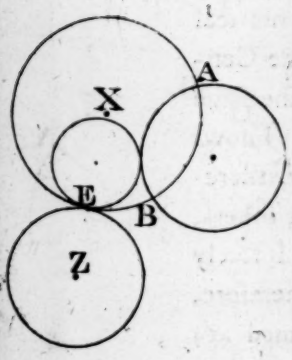
The ſix Problems are in Practical Geometry.

DEFI-

DEFINITIONS.

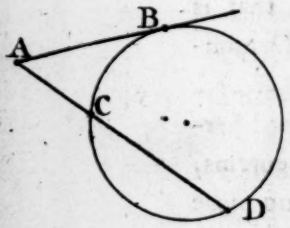
For the definition and geneſis of a Circle, ſee Def. 19. and 20, in the General Introduction.

DEF. 1. EQUAL CIRCLES are ſuch as have equal Diameters or equal Radii.



2. Circles are ſaid to cut one another, when their Circumferences croſs or interſect each other in two Points. A and B.

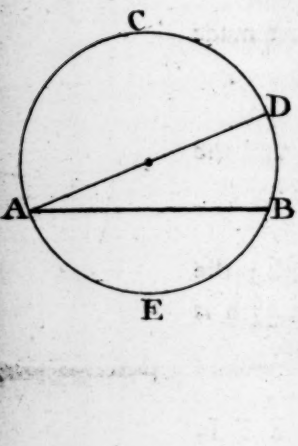
3. Circles are ſaid to touch, when their Circumferences meeting, either internally or externally, in any part, they do not cut each other.
As X and Z touching in the Point E.



4. A TANGENT. A Right Line is called a Tangent, when, being in the ſame Plane with a Circle, it touches the Circumference, only, without cuting it. As AB in B.

And, the Point B, in which it touches the Circle is called the POINT OF CONTACT.

5. A SECANT is a Right Line, drawn from any Point without a Circle cuting the Circumference in two Points. As AD, in C and D.



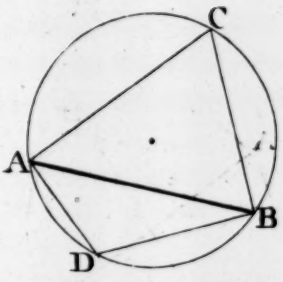
6. A SEGMENT, of a Circle, is any portion of a Circle, cut off by a Right Line; which is called a CHORD LINE, or SUBTENSE.
As AB making two Segments.

If the Segment be greater or leſs than a Semicircle (ACD or AED) it is called a GREATER or LESSER SEGMENT.

7. The Angle of a Segment, is the mixed Angle which is contained under the Chord Line and a portion of the Circumference. As ABC or ABE.

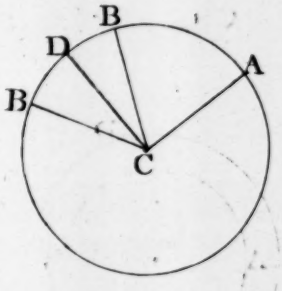
The

The Angle of a greater Segment is obtuſe, of a leſſer Segment it is acute.



8. An Angle in a Segment is that which is contained by two Right Lines, drawn from each extreme of its Baſe, or Subtenſe, to any Point in the ark of the Segment. As ACB or ADB.

9. Similar Segments are ſuch as contain equal Angles, or whoſe Angles are equal.



10. A SECTOR of a Circle, is comprehended between two Radii or Semi-diameters, and an Ark, or portion of the Circumference, intercepted between them. As AC, CB and the Ark AB.

If the Radii AC, CD contain a Right Angle, it is called a QUADRANT; as ADC.

11. An Angle in a Circle is ſaid to inſiſt or ſtand upon an Ark of the Circumference, which is oppoſite to the Angle; or that portion of the Circumference below the Chord Line.

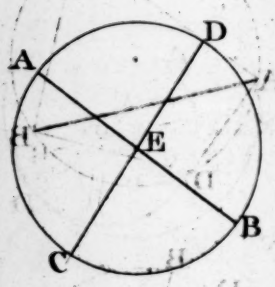
The Circumference, or curved Line which bounds a Circle, is concave towards the Center; and, externally it is convex.

AXIOMS.

The Six Axioms, or ſelf-evident Propoſitions, which follow, contain properties of a Circle which are neceſſary to be known, previous to the Demonſtration of the Theorems which follow after. I call them Axioms, becauſe they are ſelf-evident; but ſince that will not, by ſome, be allowed, arbitrarily, I have endeavoured to illuſtrate them, and to make the truth appear clear and manifeſt.

AXIOM 1st, All Lines drawn from the Center of a Circle to the Circumference are equal. As EA, EC, &c.

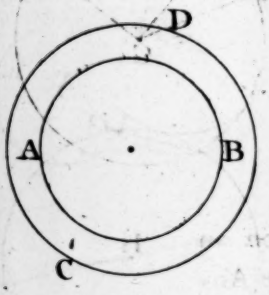
For, they are all Radii or Semidiameters.



2nd. Two or more Diameters, of a Circle, mutually biſect each other.

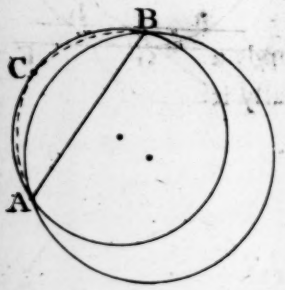
For they all paſs through the Center; conſequently, they are biſected, in the Center. As AB and CD, in the Point E.

3rd. Circles, in the ſame Plane, which cut, or touch each other, inwardly (outwardly it is maſeſt) have not the ſame Center.



For, if they have the ſame Center, and equal Radii, they muſt agree in every part. And, if they have the ſame Center, and unequal Radii, they are parallel Circles, and can neither cut nor touch each other, in any part; for they are, every where, equi-diſtant. As AB, and CD.

4th. Circumferences of Circles cut each other in two points only.



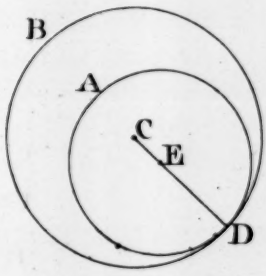
For, if it were poſſible to cut, or touch only, in three Points, A, B, & C, each of the points of Section, will be equally diſtant from the Center of each Circle; and conſequently they will have the ſame Center, which is contrary to the 3rd.

5th. A Right Line joining two Points (A and B) in the Circumference of a Circle, falls entirely within the Circle. This is manifeſt.

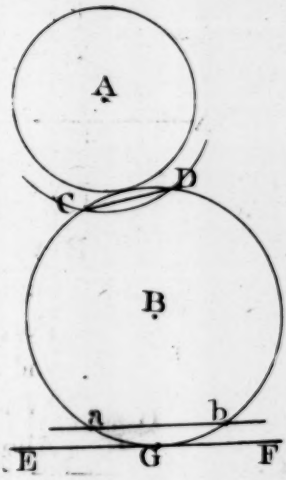
For if not, the Curve of the Circle, ACB, muſt coincide in ſome entire part, with the Line AB; which is contrary to the 19th Definition, in the General Introduction, and the Geneſis of a Circle, in N. B. Def. 20.; from which it is manifeſt, that the Curve of a Circle cannot fall in with a Right Line, in any part, it being uniform in every part.

6th. Cir-

6th. Circles touch each other, or a Right Line, in one Point only.



1. For, if the Circles, AD and BD, touched inwardly, in more than a Point, as at D, the Curve of the leſſer Circle, AD, muſt coincide in ſome part, entirely, with the curve of the larger Circle, BD, which, from the geneſis of a Circle, cannot be; ſeeing, they have unequal Radii, CD and ED; which produce different Curves, according to the Radius, which cannot, in any Part, fall into each other (for if they did they muſt coincide entirely in ſome part) therefore, they touch each other but in one Point.



2. If the Circles, A and B, touched outwardly, in more than a Point, they muſt cut each other; or their Circumferences will be a Right Line, in ſome part, as CD; which cannot be, for, it falls within both Circles; by the 5th.

3. The Circle, B, touches the Right Line, EF, but in one Point, at G.

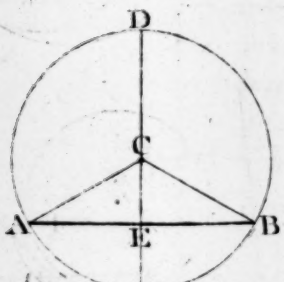
For, if it touched in more than a Point, it muſt coincide, in ſome entire part, with the Right Line EF, which cannot be (Def. 19. and 20.)

If any other Line, a b, be drawn between the Line EF and the Center, it will cut the Circle in two points of its Circumference, and the part, a b, of that Line falls within the Circle; by the laſt.

As this Property of Circles can only be known ſpeculatively, I preſume, that, what I have ſaid is as convictive, and ſatisfactory, as any Demonſtration whatever.

THEOREM I. 3. Euclid.

If a Right Line, drawn through the Center of a Circle, biſects a Chord Line, not drawn through the Center, it will cut it perpendicularly.



In the Circle ADB; let the Right Line DE paſs through C, the Center, dividing the Chord Line AB into two equal parts, at E.

I ſay, the Line DE will be perpendicular to AB.

From the Center, C, draw CA, CB.

DEM. The Triangles ACE, ECB are equilateral and equiangular to each other.

For, CE is common to both; AE=EB, - - Hyp.
and AC is equal to CB. - - - - - Ax. 1.
Wh. becauſe AC=CB the Angle CAE=CBE.- Th.9.1.
Alſo, the Angle AEC=CEB. - - - - - 8. 1.
conſequently, they are Right ones. - - - C. 2. 1. 1.
Therefore, DCE is perpendicular to AB-Def. 10. & 11. 1.

COR. 1. A Chord (AB) being biſected by another Right Line, at Right Angles with it (in E) the perpendicular Line, (DE) will paſs through the Center of the Circle.

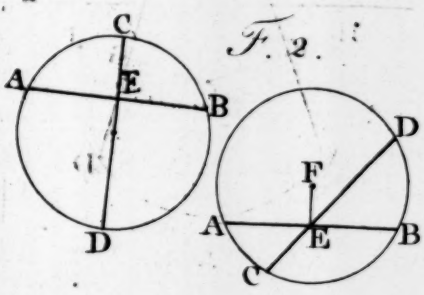
COR. 2. A Perpendicular drawn from the Center of a Circle to a Chord Line, biſects the Chord; and alſo the Ark of that Chord. As AFB, in F.

Hence, a Right Line, or an Ark of a Circle, is biſected. Prob. 8th.

THEO-

THEOREM II. 4. Euclid.

If, in a Circle, two Chord Lines cut each other, and are not both drawn through the Center, they cannot biſect each other.



Let AB and CD be two Chord Lines cuting each other, in E.

If one of them, CD (Fig. 1.) paſſeth through the Center, it is evident, that it cannot be biſected by the other, AB, which does not paſs through the Center.

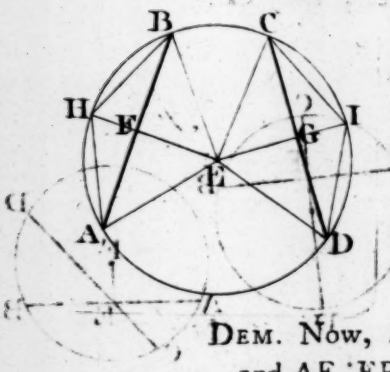
If neither of them paſſes through the Center, one of them may be biſected, but both cannot.

Dem. For, if AB (Fig. 2.) is biſected, in E, a Right Line, FE, joining the Center, F, and the point of Section, E, will be perpendicular to AB. - - - - - - Th. 1.
conſequently, AEF & FEB are Right Angles. - Def. 10. 1.
And, if CD be alſo biſected, in E;
FE is, alſo, perpendicular to CD. - - - - - Th. 1.
Wherefore, CEF and FED are Right Angles. - - Sup.
But, AEF, FEB are Right Angles. - - - proved.
conſequently, CEF, FED are not Right Angles. - Ax. 2. 1.
for it is evident, that one is greater and the other leſs.
Therefore, CD is not biſected in E. Q. E. D.

THEO-

THEOREM III. 14. Euclid.

Equal Chord Lines, in a Circle, are equally distant from the Center; and Chord Lines which are equi-distant, are equal.



Let AB and CD be equal Chord Lines in the Circle ADB.

I say, they are equally distant from the Center, E.

From E, the Center, draw EF and EG perpendicular to AB and CD; and join AE, EB, EC, and ED.

DEM. Now, AB is equal to CD, by the Hypothesis.
and AE, EB, EC, and ED are all equal. - - Ax. 1.
wh. the Triangles AEB, CED are congruous. - Def. 44.
(for, they are equilateral, to each other, by Construction)
consequently, they are equiangular, - - - - 7. 1.
And, being also Isosceles, the Angles at A and B, C and D, are all equal amongst themselves. - - - 9. 1.

Now, since AB & CD are bisected in F & G, AF=DG.
Wherefore, in the Triangles AFE, EGD, the Sides, AE, AF, are equal to ED, DG, respectively;
and they contain equal Angles, FAE=EDG. - proved.
Therefore, EF is equal to EG. - - - - - - 8. 1.

2nd. AB and CD are two Chord Lines, equally distant from the Center of the Circle ADB. I say, AB=CD.

Draw the equal Lines, EF, EG. perpendicular to AB and CD; which will bisect them, in F and G.- Cor. 2. 1.
Draw AE and ED.

DEM. Because AE=ED, AE□=ED□; &, EF□=EG□.
But, AE□=AF□+EF□; & ED□=EG□+GD□.
conf. AF□+EF□=EG□+GD□; wh. AF□=GD□
Therefore, AF=GD (Ax. B 2.) & conf. AB=CD-Ax. 5. 1.

The ſecond Part of this is the converſe of the former; which, I ſhould have made a Corollary to it; but on account of the different manner of Demonſtration; by the laſt, of which, Euclid demonſtrates both.

Cor. 1. Equal Chord Lines, in the ſame or equal Circles, ſubtend equal Angles at the Center.

For, ſince AB=CD, and AE, EB, EC, and ED are all equal, the Triangles AEB, CED are congruous; conſ. the Angle AEB=CED. - - - - 7. 1.

And, ſince equal Circles have equal Radii, the Angles which equal Chords ſubtend, in equal Circles, are alſo equal.

Cor. 2. Equal Chords, in the ſame or equal Circles, ſubtend equal Arks, and cut off equal Segments. And, equal Arks have equal Chords.

AB=CD; and, ſince EF=EG, if EF and EG be produced, to H and I, FH=GI. - - - - Ax. 7. 1.
And, ſince EF and EG are perpendicular to AB and CD, the Arks AHB, CID, are biſected, in A and I. - C. 2. 1.
wherefore, the Triangle AHB=CID;
the Arks, AH, HB, CI, and ID, are all equal, and the whole Ark AHB is equal to CID. - - Ax. 5. 1.

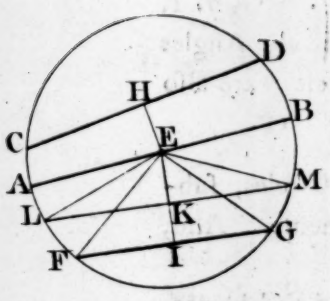
For, if AB was applied to CD, being equal, they would perfectly agree; the Point A with D, and B with C; alſo, the Point H would agree with the Point I, and every other Point, in the Ark AHB, with a correſponding Point, in the Ark CID; conſequently, the whole Ark, AHB, would coincide with the Ark CID; therefore the Segment AHB=CID; and, conſ. DAHC=ADIB.

The ſame may be ſaid of equal Circles, having equal Radii.

THEO-

THEOREM IV. 15. Euclid.

A Diameter is the greateſt Right Line, which can be drawn in a Circle; and, of all other Chord Lines, that is the greateſt which is neareſt to the Center.



Let AB be a Diameter, and E the Center of the Circle CDG.

CD and FG are Chord Lines, at different diſtances from the Center.

I ſay, that AB is the greateſt; and FG, the fartheſt off, is the leaſt of the three.

Draw the Perpendiculars EH and EI, to CD and FG, from the Center.

DEM. Becauſe FG is farther from the Center than CD, EI is greater than EH. (3.) Make EK equal te EH. Draw KL, perpendicular to EK, and produce LK to M; and, laſtly, draw EL, EM, EF, and EG.

Now, becauſe EH is equal to EK, and are perpendicular to CD and LM; LM is equal to CD. - Th. 3.
But AE+EB (eq. LE+EM) is greater than LM. - 13. 1.
Therefore, AB is greater than LM; i. e. than CD, eq. LM.

2nd. In the Triangles LEM, FEG, the two Sides LE, EM, are equal to the two Sides FE, EG, and the Angle LEM is greater than FEG.

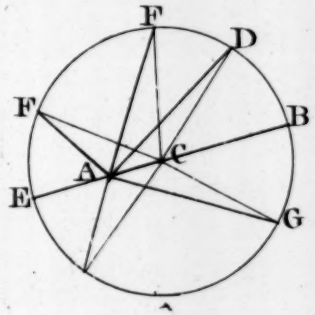
Therefore, FG is leſs than LM; i. e. than CD, which is equal to LM. - - - - - - Cor. to 8. 1.

THEO-

THEOREM V. 7. Euclid.

If any Point, except the Center, be taken within a Circle, and, from that point, divers Right Lines be drawn to the Circumference; the greateſt of thoſe Lines is that which paſſes through the Center; and, the leaſt is the remainder of that Line, produced to the Circumference.

Of all others, drawn from that Point, that which falls neareſt to the Line, paſſing through the Center, is greater than the more remote ones; and, but two equal Right Lines can be drawn, from that Point, to the Circumference.



Let A be the Point aſſumed, in the Circle EFG; from which draw AB, AD, AE, &c.

Firſt; AB, which paſſes through the Center, C, is greater than AD, or AF, or any other Line, which can be drawn from the point A. Draw DC.

Dem. In the Triangle ADC, the two Sides, AC, CD, are greater than the remaining Side, AD. - - 13. 1.
But, CD is equal to CB, - - - - Ax. 1.
wherefore, AC + CD = AC + CB, i. e. AB. - Ax. 6. 1.
Therefore, AB is greater than AD.

2nd. AE is the ſhorteſt Line, which can be drawn from the Point A. Draw any Line, AF, and join FC.

Then, in the Tri. AFC, AF + AC is greater than CF.
But, CF is equal to CE, - - - - Ax. 1.
Wh. (taking away AC) AF is greater than AE - Ax. 8. 1.

3rd. That Line is the longeſt, which falls neareſt to AB.

In the Triangles ADC, AFC, AC, CD are equal to AC, CF, reſpectively; for AC is common, & CD=CF-Ax. 1. But, the Angle ACD is greater than ACF. - Ax. 2. 1. Therefore, the Side AD is greater than AF.- Cor. to 8. 1.

4th. No more than two equal Lines can be drawn, from the Point A, to the Circumference.

For, ſince AF is proved leſs than AD, and AB greater, every Line which can be drawn, from A, between D and F, will be leſs than AD, and between B and D greater; wherefore, no other equal Line can be drawn on that Side EB.

Make the Angle BCG=BCD; (Pr. 4.) and draw AG. Then, becauſe the Angle BCG=BCD, GCA=ACD. For, BCG+GCA=BCD+DCA. - Th. 1.1. & Ax. 9. And, becauſe CG=CD, and AC is common; we have AC, CG, reſpectively equal to AC, CD; and the Angle ACG=ACD; therefore AG=AD - 8. 1.

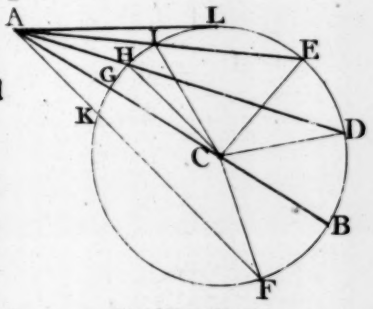
And, as no other Line, equal to AG, can be drawn on that Side BE; conſequently, no more than two equal Lines can be drawn from the ſame Point, A, to the Circumference.

Cor. Hence it is evident, that, if from any Point in a Circle, more than two equal Lines can be drawn to the Circumference, that Point is the Center.

THEOREM VI. 8. Euclid.

If from any Point without a Circle, Right Lines are drawn to the concave part of the Circumference; that which paſſes through the Center, is the greateſt; and that Line which falls neareſt to it, is greater than that which is more remote.

Of thoſe, which fall upon the convex part of the Circumference; that is the leaſt, which, if produced would paſs through the Center; and that Line, which falls neareſt to it, is leſs than any other, further off. And, no more than two equal Lines, can be drawn from any Point, without the Circle, either to the convex or concave part of the Circumference.



Aſſume the Point, A, at pleaſure, and draw AB, AD and AE.

Firſt; AB which paſſes through the Center, C, is greater than AD or AE, not paſſing through the Center. Draw CD.

Dem. Then, in the Triangle ACD, the two Sides AC, CD are greater than AD. - - - - - - Th. 13. 1.
But, CD=CB; wherefore, AC+CB, i. e. AB, =AC+CD; conſequently, AB is greater than AD.

2nd. AD is greater than AE. Draw CE.

Then, in the Triangles AEC, ADC, the Sides AC, CE, are reſpectively equal to AC, CD. - - - Ax. 1.
and the Angle ACD is greater than ACE. - - Ax. 2. 1.
therefore, AD is greater than AE. - - Cor. to 8. 1.

3rd. AG is leſs than AH or AI. Draw CH and CI.

Then, in the Tri. ACH; AH + HC is greater than AC. But, GC is equal to HC; - - - - Ax. 1. wherefore, AC—GC, i. e. AG, is leſs than AH - Ax. 8. 1.

4th. AH is leſs than AI, which falls further from AC.

For, in the Tri. AIC, the two Sides, AI, IC, are greater than AH, HC, to any Point, H, within the Triangle - 14. 1. But IC = HC; wherefore, AH is leſs than AI. - Ax. 8. 1.

5th. No more than two equal Lines can be drawn from any point (as A) without the Circle, either to the convex or concave part of the Circumference.

For, ſince all Lines drawn from A, between G and H, are leſs than AH, and between H and I, greater; no other Line, drawn on that Side AB, can be equal to AH.

But an equal Line, AK, may be drawn on the other Side.

So likewiſe; all Lines, drawn from A to the concave Periphery, between AB and AD, are greater than AD; and between AD and AE, leſs; conſequently, no other Line can be drawn, on that Side AB, equal to AD.

But, if BF be made equal BD, and AF be drawn, AF is equal to AD. Draw CF; which is equal to CD.

For, the Angle FCB = BCD, by Conſtruction, wherefore, FCA = ACD - - Th. 1, 1. & Ax. 7. 1. Wh. in the Triangles ADC, ACF, the Sides AC, CD are reſpectively equal to AC, CF; for CF, CD are Radii; and, the Angle ACD = ACF, therefore AF = AD - 8. 1.

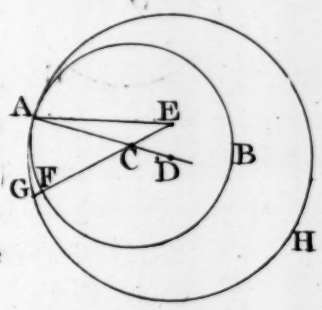
No other Line, equal to AF (eq. AD) can be drawn on that Side of AB; wherefore, but two equal Lines can be drawn from any Point, A, without the Circle, either to the convex or concave part of the Circumference.

Cor. The greateſt Right Line that can be drawn, from a Point without a Circle, to the convexity of the Circumference, is equal to the leaſt drawn to the concave part.

For they unite in a Tangent to the Circle, at L.

THEOREM VII. 11. 12. Euclid.

If two Circles touch each other, a Right Line, joining their Centers, will paſs through the Point of contact of the two Circles.



Firſt; let AB and AH be two Circles, touching each other, inwardly, in A.
C is the Center of the leſſer Circle, AB.

Draw the Right Line AC; it will, if produced, paſs through D, the Center of the other Circle.

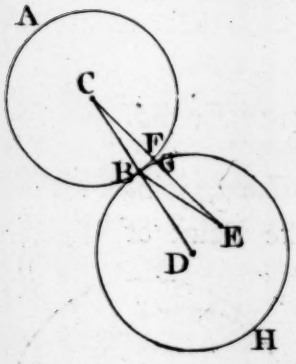
If not; let any other Point, as E, be the Center of AH, and draw EC, cuting the two Circles in F & G, and join AE.

Dem. Then, becauſe C is the Center of the Circle AB,
CA is equal to CF. - - - - - Ax. 1.
If CE be added to both, then CE+CF=CE+CA - 6. 1.
But, CE+CA is greater than AE. - - - Th. 13. 1.
and, EA is equal EG (by the Suppoſition). Ax. 1.
wherefore, EG is leſs than EF (the greater than the leſs) which is abſurd, and cannot be.
Therefore, E is not the Center of the Circle AH;
conſequently D is the Center; and, the Right Line DC paſſes through the Point of contact, A.

2ndly. AB and BH are two Circles, touching each other, outwardly, at B; C is the Center of AB.

Draw the Right Line CB; which, if produced, will paſs through the Center, D, of the other Circle.

If it be denied, let E be ſuppoſed to be the Center of BH, and draw CE and BE.



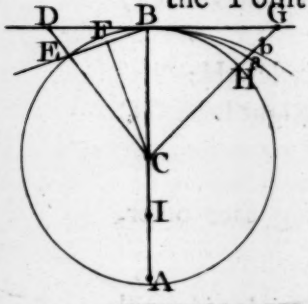
Then, becauſe C is the Center of the Circle AB, CF is equal to CB;
and, if E be the Center of BH, EG=EB.
Wherefore, CF+GE=CB+BE. - Ax. 6. 1.
And, if FG be added to the former; CE, i. e. CF+FG+GE, is greater than CB+BE.
But, CB+BE is greater than CE - Th. 13. 1
wherefore, BC+BE is both greater and leſs than CE, which is abſurd, and cannot be.

Therefore, E is not the Center of the Circle BH;
conſequently D is the Center: and, the Right Line, CD, joining the two Centers, C and D, paſſes through B, the Point of Contact.

Cor. Circles touch each other, either inwardly or outwardly, in a Point only.

THEOREM VIII. 16. Euclid.

If a Right Line be drawn, through the extreme point of a Diameter of a Circle, at right angles with the Diameter, it will fall wholly without the Circle; and no other Right Line can be drawn, from the Point of contact, between the Tangent and the Circle.



Let AB be a Diameter, and DG a Right Line drawn at right angles with AB, through the extreme B.

Draw CD cuting D in DG.

I ſay, the Point D is without the Circle.

Dem. In the Tri. CDB, the Angle DBC is a right one;
conſ. BDC is acute; wh. DC is greater than BC. - 12. 1.
But, the Point B, is in the Circumference.
wherefore, the point D is beyond the Circumferenee - Ax. 1.
conſequently BD falls wholly without the Circle.

Or thus after Euclid.

If BD does not fall without, it will cut the Circle; ſuppoſe in B and E. Draw EC.

Then, in the Triangle EBC, EC=CB; - Ax. 1.
wherefore, the Angle CEB=CBE.
But, CBE is a right angle (Hyp.) conſ. CEB is a right one.
And the two angles, CEB, EBC, of the Triangle CEB, are equal to two Right Angles.
But all the three Angles of every Triangle=two R. angles.
Therefore, BE, i. e. BD, does not cut the Circle, but muſt neceſſarily fall without it.

2nd. Let BE be drawn (if poſſible) between DB and the Circumference of the Circle.

Now, ſince CBD is a Right Angle, CBE is acute.
Let CF be drawn perpendicular to BE.
Then, CF, ſubtending an acute Angle, is leſs than CB.
But, the Point B is in the Circumference.
conſ. CF does not reach the Circumference. - Ax. 1.
wherefore, the Point F is within the Circle.
Therefore, the Right Line BE cannot be drawn between the Tangent, BD, and the Circumference.

Cor. 1. A Tangent touches a Circle in one Point only. For if it touched in two Points it would cut the Circle.

Cor. 2. A Right Line perpendicular to the Tangent, at the Point of Contact, paſſes through the Center.

Cor. 3. A Right Line drawn from the center of the Circle to the Point of contact, of a Right Line touching the Circle, is perpendicular to that Tangent.

Cor. 4. The Angle of a Semicircle is a right one.
For, it is greater than any right-lined acute angle.

N. B.

N. B. The external Angle, made by a Tangent and the adjoining Ark of the Circle, is leſs than any right-lined Angle whatever.

For it has been proved, that a Right Line cannot be drawn, from the Point of contact, between the Tangent and the Circumference; wherefore, the Angle GBH cannot be made leſs by a Right Line, ſeeing it will cut the Circle, and conſeqnently make a larger Angle, at ſome other Point in the Circumference.

Yet may the Angle GBH be leſſened infinitely, by curved Lines; which is, apparently, a Paradox.

For ſince the Angle of a Semicircle is Right, and the Tangent BG is perpendicular to the Diameter, AB, there cannot be left any remainder of the Angle as a Compliment to it; becauſe, the Compliment of an Angle is its deficiency of a Right Angle; or (being obtuſe) to two Right Angles.

Now, ſince it has been proved that a Right Line cannot touch a Circle but in one Point only; it muſt hold equally true, in reſpect of a large Circle as of a ſmall one.

Wherefore, if any other Radius, greater than BC, be taken, as BI, and an Ark, Ba, be drawn, it is evident that the Angle GBa is leſs than GBH.

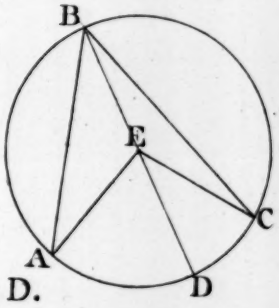
If a larger Radius, BA, be taken, and, on the Center A, an Ark, Bb, be drawn, the Angle GBb is leſs than GBa; notwithſtanding, the Angles, ABH, ABa, and ABb, are the ſame (by the 4th Cor.) and it is evident, that if a ſtill larger Radius be taken, the Angle GBb may ſtill be leſſened infinitely.

SCHOL. *Hence it is manifeſt, that any Right Line,* GH, *is infinitely diviſible, by enlarging the Radius* BA *infinitely; ſeeing that, the Circumference of a Circle cannot coincide with two Points,* B *and* G, *of the Right Line* BG.

THEO-

THEOREM IX. 20. Euclid.

An Angle, at the Center of a Circle, is double of an Angle touching the Circumference, when both ſtand on the ſame Ark.

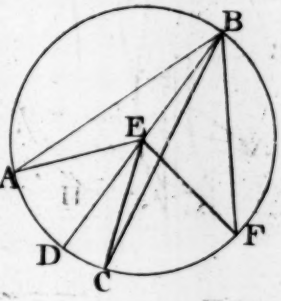


In the Circle ABC, let AEC be an Angle at its Center, and ABC is an Angle touching the Circumference; both ſtanding on the ſame Ark AC.

I ſay, the Angle AEC is double the Angle ABC.

Draw the Right Line BE, and produce it to D.

DEM. In the Triangle ABE, becauſe the ſide AE = EB,
the Angle A is equal to ABE - - - - - - - 9.1.
But, the external Angle AED = the Angle A + ABE - 10.1
Therefore, the Angle ABD = half the Angle AED
Alſo DBC = ½DEC; for the Triangle EBC is Iſoſceles.
Conſ. AED + DEC (i. e. AEC) = twice ABD + DBC;
i. e. equal to twice ABC.
Therefore, the Angle AEC is double the Angle ABC.



Caſe 2nd. When the Angle CEF, at the Center, falls without the Angle CBF, at the Circumference. Draw BED, as before.

Now, the external Angle, DEC, = ECB + EBC; (10. 1.) and ECB = EBC - 9.1 wherefore, DEC is equal to twice DBC.
And, the Angle DEF = EBF + EBF;
therefore. equal twice DBF.
Conſequently, DEF - DEC, i.e. CEF, = 2DBF - 2DBC, i.e. 2CBF. Therefore, the Angle CEF is double CBF.

COR. 1. The Angle at the Circumference, ſtanding on any Ark, is equal to an Angle at the Center, on half that Ark.

For, if the Ark AC be=CF, the Angle AEC=CEF.* But, the Angle AEC is double ABC, &, CEF double CBF conſequently the Angle AEC (eq. CEF)=ABC+CBF; i. e. the Angle ABF, on the Ark ACF, is equal to AEC, (or CEF) on AC, or CF, half that Ark.

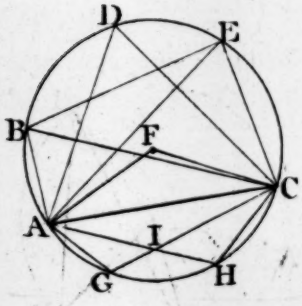
COR. 2. In the ſame or equal Circles, Angles ſtanding on equal Arks, whether they be at the Center or at the Circumference, are equal to one another.

This is evident from the laſt; for the Ark AC being equal to CF, the Angle AEC=CEF; and, the Angle ABC (eq. half AEC)=CBF, half CEF.

And, becauſe the Radii of equal Circles are equal, the Angles on equal Arks, in equal Circles, are alſo equal. And converſely, equal Angles ſtand on equal Arks.

THEOREM X. 21. Euclid.

All Angles, which ſtand on the ſame Ark, or are in the ſame Segment of a Circle, and touching the Circumference, are equal to one another.



In the Segment ADC, the Angles ABC, ADC, and AEC, are all equal.

To the Center, F, draw AF and CF, making the Angle AFC.

DEM. Then, the Angle AFC, at the Center, is double AEC, at the Circumference. For, they ſtand on the ſame Ark, AGC, or Chord, AC. But the Angle AFC is double ABC, or any other Angle, ADC, touching the Circumference. - - - - Th. 9. Wherefore, the Angles ABC, ADC, &c. being each equal half AFC, are, therefore, equal amongſt themſelves.

* Theory of Plane Angles, Art. 4. Page 23.

If the Segment be a Semicircle, or leſs than a Semicircle, it may be demonſtrated after this manner.

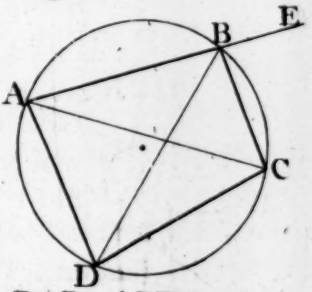
AGC and AHC are Angles in a leſſer Segment, AGHC. The Angle AIC (being external, in reſpect of the Triangles GAI & ICH) = the Angle G + GAI; and alſo, to H+HCI; wherefore, the Angle G+GAI=H+HCI. But, the Angle GAI, i.e GAH=HCI, i.e. HCG; above. Therefore, the Angle AGC=AHC. Q. E. D.

Cor. Right Lines, BC and AH, joining the extreme Points of two equal Arks, AB and CH, in the Circumference of a Circle, ſo as not to croſs each, are parallel.

For, having drawn AC) the Arks AB, CH being equal. the Angle BCA is equal to the Angle CAH. - C. 2. 9
Therefore, BC is parallel to AH. - - - - Th. 4. 1.

THEOREM XI. 22. Euclid.

In every Quadrilateral, inſcribed in a Circle, the oppoſite Angles are equal to two Right Angles.



ABCD is a Trapezium incribed in a Circle.

I ſay, the Angles A and C, alſo B and D, are, together, equal to two Right ones.

Draw the Diagonals AC and BD,

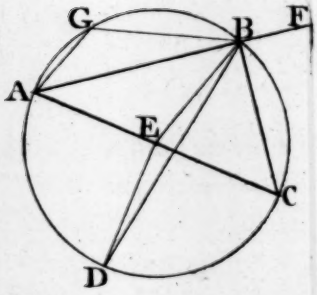
Dem. Then, in the Triang. ABD, the Angle DAB+ABD added to BDA are equal to two Right ones. - - 10. 1.
But, the Angle BCA = BDA; and ACD = ABD. - 10.
Wh. BCA + ACD, i. e. BCD, + BAD = two Right Angles

Cor. 1. The external Angle, CBF, made by producing any Side, AB, of a Quadrilateral inſcribed in a Circle, is equal to the oppoſite Angle, ADC. - by Theo & 1. 1.

2. Every Parallelogram inſcribed in a Circle is a Rectangle.

THEOREM XII. 31. Euclid.

An Angle, at the Circumference, in a Semicircle, is a Right Angle.



In the Circle ABCD, let AC be a Diameter, and E the Center.

To any Point, B, in the Circumference, draw AB and CB.

I say the Angle ABC is a Right Angle.

Draw EB; the Triangles, AEB, BEC, are Iſoſceles. AE, EB, and EC being equal.

DEM. Then, becauſe AE=EB, the Angle EAB=ABE and, becauſe EB=EC, the Angle EBC=BCE - 9. 1. wherefore, the Angle ABC=BAC+ACB - - Ax. 6. But, if one Angle of a Triangle be equal to the other two, it is a Right Angle. - - - - - C. 1. 10. 1. Therefore, ABC is a Right Angle Q. E. D.

Or, if AB be produced, the external Angle CBF is equal to BAC + BCA - - - Th. 10. 1.

It may be otherwiſe proved, after this manner.

Draw ED perpendicular to AC, and join BD.

Now, the Angle AED is double the Angle ABD. And the Angle DEC is double the Angle DBC. - Th. 9. But, the Angles AED, DEC, are Right ones. - Con. wherefore ABD, DBC are each half Right - C. 1. 10. 1. Conſ. ABC, i. e. ABD+DBC, is a Right Angle.

Or, the Angle ABC at the Circumference, ſtanding on the Ark or Semi-circumference ADC, is equal to an Angle at the Center, AED or DEC, (which are Right Angles) on half that Ark. - - - - - - - Th. 9.

COR.

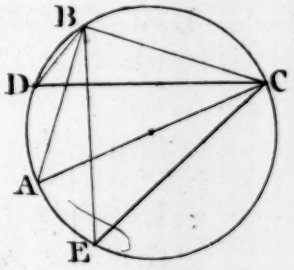
From this Theorem is deduced that moſt elegant and expeditious Method for making a Right Angle, or drawing a perpendicular at the extremity of a Right Line; Prob. 10. Method. 1ſt.

Cor. An Angle in a Segment, leſs then a Semicircle, is obtuſe; and an Angle in a greater Segment is acute.

In the Segment ADCB ſeeing that the Angle ABC is right, ACB, in that Segment, is acute. - - - - C3. 10. 1.
And, the Angle AGB+ACB=two Right Angles - Th. 11.
But ACB is acute; wherefore AGB is obtuſe. - Def. 13.

Theo. The Angle in a leſſer Segment exceeds, and the Angle in a greater, is deficient of a Right Angle, by the Angle made between the Chord of the Segment and a Diameter, drawn from either extreme of the Chord.

Let AC be a Diameter, and ABC is a Right Angle.
Draw two Chords, DC and EC, and join DB and EB.



Then, in the leſſer Segment DBC, the Angle DBC is greater than the Right Angle ABC, in a Semicircle. - Ax. 1.
And the difference is the Angle DBA,

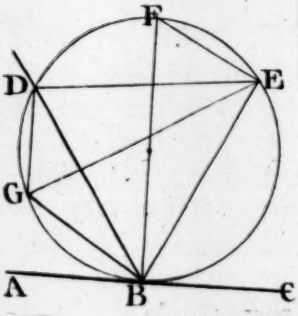
But, the Angle DBA=DCA.-Th. 10.
Therefore, the Angle DBC, in a leſs Segment, exceeds a Right Angle, by the Angle ACD.

2nd. In the greater Segment EDBC, the
Angle EBC is leſs than a Right Angle ABC.
And, the deficiency is the Angle ABE, equal ACE.
Therefore, the Angle, EBC, in a greater Segment, &c.

THEO.

THEOREM XIII. 32. Euclid.

The Angle made between a Tangent and a Right Line, cuting the Circle, drawn from the Point of Contact, is equal to the Angle made in the oppoſite Segment.



Let ABC be a Right Line, touching the Circle BDE, in B.

Draw at pleaſure, from B, the Chord BD.

The Angle ABD is equal to DEB, or any Angle made in the Segment DBE.

And the Angle DBC is equal to DGB, in the other Segment, DBG.

Draw BF perpendicular to AC, and join FE.

DEM. Now BF is a Diameter of the Circle. - - C. 2. 8.
wherefore, FEB is a Right Angle. - - - - Th. 12.
But, ABF is a R. Angle (Con.) conſ. FEB=ABF, Ax. 9.
But, the Angle FED=FBD (10) Th DEB=DBA, Ax. 7.

2nd. Let BG be the Chord of the Segment GFEB. Draw GE, making an Angle, GEB.

Then, the Angle GDB (eq. GEB)=GBA; as above.
And, the Angle G+GDB+DBG=two R. Angles. - 10. 1.
But, the Angle ABD+DBC= two Right Angles. - 1. 1.
and the Angle GDB (eq. GBA)+DBG=DBA. - Ax. 2. 1.
Therefore the remaining Angle, DGB=DBC. - Ax. 7. 1.

Or, in the Quadrilateral DEBG, the oppoſite Angles, DGB, DEB, are equal to two Right Angles. - Th. 11.
And the Angle ABD+DBC=two Right Angles. - 1. 1.
But, the Angle DEB=DBA, (as above) Th. DGB=DBC.

THEO-

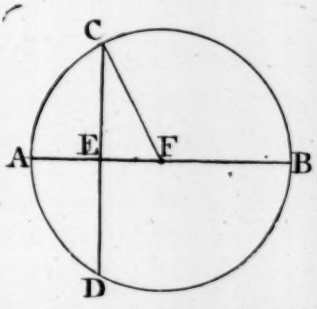
THEOREM XIV. 35. Euclid.

In Circles, if two Chord Lines cut each other; the Rectangle, contained under the Segments of one Line, will be equal to the Rectangle under the Segments of the other.

1ſt. When both paſs through the Center, it is evident.

For, the Rectangle under the Segments of each, is the Square of the Radius; conſequently they are equal.

2nd. In the Circle ACBD; if AB, paſſing through the Center, biſects CD, which does not paſs through the Center, in E; then, the Rectangle AEB, is equal to the Square of CE, or ED.

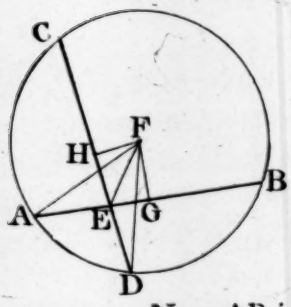


Let F be the Center. Draw CF.

Dem. Then, becauſe AB is biſected, in F, and cut unequally, in E; $AE \times EB + EF\square = FB\square$ - - 5. 2.
i. e. $= CF\square, = CE\square + EF\square$. - - - - - 20. 1.
for, CEF, is a R. Angle (C. 2. 1.) and CF = FB. - Ax. 1.
Wherefore, $AE \times EB + EF\square = CE\square + EF\square$.
Take away from both, $EF\square$, which is common;
There remains $AE \times EB = CE\square$. i. e. $\square AEB = \square CED$.

3rd. When

3rd. When, neither, of them paſſes through the Center, and, when neither is biſected, by the other.



From the Center, F, draw FG and FH perpendicular to AB and CD; and join AF, FD, and EF.

Now AB is biſected in G, and cut unequally in E;
wherefore, AE × EB + EG □ = AG □ - - - 5.2.
Add, on both ſides, FG □;
then, AE×EB+EG □+FG □, = AG □+FG □ - Ax.6.1.
But EF □ = EG □ + FG □; & AF □ = AG □ + FG □ - 20.1
Therefore, AE × EB + EF □ = AF □.

After the ſame manner, it may be proved, that the Rectangle CED + EF □ is equal to AF □.

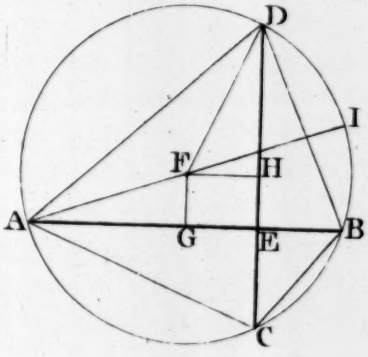
For, CD is alſo biſected, in H, and unequally cut, in E.
Wherefore, CE×ED+HE □ = HD □. Add HF □ to both.
Then, CE × ED + HE □ + HF □ (equal EF □) 20.1
is equal to HD □ + HF □; i. e. equal FD □.
But FD = AF; wherefore, FD □ = AF □ - Ax. 1.
conſequently, CE×ED+EF □ = AF □; (equal FD □)
But, it was proved, that, AE×EB+EF □ = AF □.
Wherefore, taking away EF □, from both,
there remains AE×EB = CE×ED. - - - - Ax. 7. 1.
Or, the Rectangle AEB, equal CED. Q. E. D.

THEO-

THEOREM XV.

If two Chord Lines, interſect at right angles; the four Squares, of the Segments of thoſe Chords, will be equal to the Square of the Diameter.

And, the Squares of the two oppoſite Sides of a Quadrilateral, formed by Right Lines, joining the extreme Points of the Chords, are alſo equal to the Square of the Diameter.



Let the Chords AB and CD cut each other perpendicularly, in E.

Then, the Squares of AE, EB, EC, and ED, are equal to the Square of AI, the Diameter of the Circle.

From the Center, F, draw FG and FH parallel to the Chords CD and AB; and join AF, and FD.

Dem. Now, ſince AB & CD cut each other perpend. - Hyp. and, FG, FH, are reſpectively parallel to CD & AB - Con. FG and FH are penpendicular to AB & CD. - C.2.4.1. Then, AB is biſected, in G. (1.3.) and cut unequally, in E. wherefore, $AE\square + EB\square = 2AG\square + 2GE\square$; - 9. 2. alſo $CE\square + ED\square = 2EH\square + 2HD\square$ - - - - ſame Wh. $AE+EB+CE+ED\square = 2AG+2GE+2EH+2HD\square$ But GFHE is a Parallelogram, by Conſtruction. wherefore, FH = GE, and FG = EH - - - - - 15. 1. Conſ. the Squares of the four Segments, AE, EB, CE, & ED are eq. to the Squares of AG, GF, FH & HD, twice taken. But, $AF\square = AG\square + GF\square$; $FD\square = FH\square + HD\square$ - 20.1 conſ. $2AF\square + 2FD\square = 2AG + 2GF + 2FH + 2HD\square$ Wherefore, $AE + EB + CE + ED\square = 2AF\square + 2FD\square$; i.e. $= 4\,AF\square$; - for, AF is equal to FD.

But four times AF □ = AI □; i. e. of the Diameter. - 4.2
Th the Squares of the four Segments, AE, EB, CE, & ED, are equal to the Square of the Diameter.

2nd. Having joined the extremes AD, DB, AC, and CB. Then, the Squares of AD and CB, together, are equal to the Squares of AC and DB together.

For, AD □ = AE □ + ED □; and, CB □ = CE □ + EB □
But, AC □ = AE □ + EC □; and DB □ = DE □ + EB □
therefore, AD □ + CB □ = AC □ + DB □
consequently, each is equal to the Square of the Diameter.

This extraordinary property of the Circle is otherwise demonstrated by Mr. Stone; which, notwithstanding it is indisputably true, does not carry with it that positive conviction; in so much, that, a young Geometrician would be somewhat at a loss to perceive it. —— It is as follows.

Because, the Angle AED, is a Right one, the Angles EAD, ADE, are equal to a Right Angle (32.1.) wherefore the Arks, AC; DB, on which they stand, are, together, equal to half the circumference of a Circle (26.3.)

Therefore, because the Angle in a Semicircle is a Right Angle, the Squares of AC and DB are equal to the Square of the Diameter (47.1.) And, because the Angles at E are Right Angles, the Squares of AE and EC are equal to the Square of AC, and the Squares of DE, EB, equal to the Square of DB. Therefore, the Squares of AE, EC, DE, & EB, are equal to the Squares of AC & DB; which, were proved equal to the Square of the Diameter. Therefore those four Squares, are equal to the Square of the Diameter.

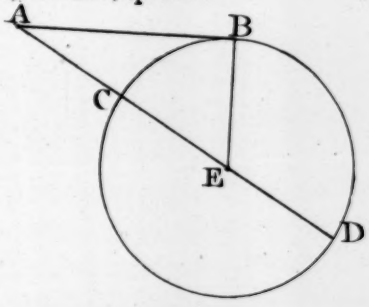
The References are, in this Demonstration, according to Euclid.

SHOL. *It is worthy of observation, that, as, in a Semicircle, the Sides containing the Right Angle are two Chords perpendicular to each other, and are equal to the Square of the Diameter; so the Squares of the Segments of all Chords which cut each other at right angles, are equal to the Square of the Diameter.*

THEO-

THEOREM XVI. 37. Euclid.

If, from any Point without a Circle, two Right Lines be drawn, the one a Tangent, the other a Secant (i. e. one touching the Circumference, the other cuting it twice) the Rectangle, under the whole Secant and the external Segment (between the Point, aſſumed, and the Circle) will be equal to the Square of the Tangent.



A is the aſſumed Point.

Let AB, touch the Circle, CBD, in the Point B; AD is a Secant, cuting the Circumference in two Points, C and D.

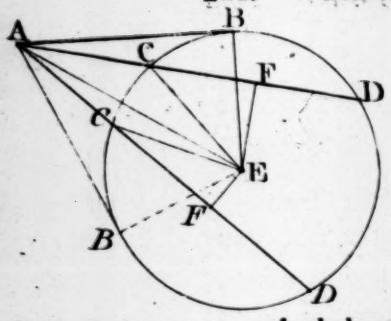
Then, the Rectangle DAC is equal to the Square of AB.

Firſt, when AD paſſes through E, the Center of the Circle.

Draw EB, from the Center to the Point of contact.

Dem. Now, becauſe CD is biſected, in E, and CA is added to it; $AD \times AC + CE\,\square = AE\,\square$ - - 6. 2
and, $AE\,\square = AB\,\square + BE\,\square$. - - - - - - 20. 1
For, ABE is a Right Angle - - - - - C. 3. 8
wherefore, $AD \times AC + CE\,\square = AB\,\square + BE\,\square$.
But, $BE = CE$; conſequently $BE\,\square = CE\,\square$. - Ax. B. 2
wherefore, $AD \times AC + CE\,\square = AB\,\square + CE\,\square$.
Take $CE\,\square$ from both, there remains $\square DAC = AB\,\square$.

2nd. When the Secant falls on either ſide of the Center.



From the Center, E, draw EF perpendicular to the Secant, AD, or A*D*, and join the Points CE and EB.

Then, becauſe EF is perpendicular to CD, CD is biſected in F. - - C. 1.1

And, becauſe CD is biſected, in F, and AC is added;

AD × AC + CF □ = AF □ (6. 2.) Add EF □ to both;

Then, AD × AC + CF □ + EF □ = AF □ + EF □ - Ax. 6.1

But, CF □ + EF □ = CE □;

and AF □ + EF □ = AE □. - - - - - Th. 20. 1

wherefore, AD × AC + CE □ = AE □;

i. e. equal to AB □ + BE □. - - - - - - 20. 1

But, BE = CE; wh. AD × AC + BE □ = AB □ + BE □.

And, if BE □ be taken from both,

there is left AD × AC = AB □. Q. E. D.

Cor. Hence, it is evident, that the Rectangles under every Right Line, cuting a Circle, from the ſame Point without the Circle, and the external Segment, are equal.

The Rectangle DAC = *DAC*; for, they are each equal the Square of the Tangent, AB.

2nd. If from the ſame Point A, without a Circle, two Right Lines AB, A*B*, are drawn, to the Circle, one on each ſide, touching the Circumference, thoſe two Tangents are equal.

For, AB □ = A*B* □, being each equal to the Rect. DAC

But, equal Squares have equal Sides;

conſequently, AB is equal to A*B*. - - - Ax. B. 2

3rd. If

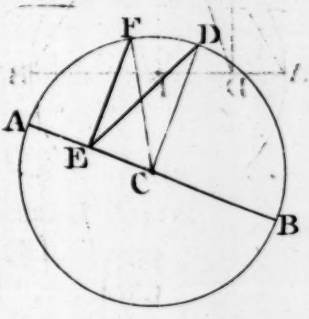
3rd. If from any Point, without a Circle, two Right Lines are drawn to the Circle; one of which Lines cuts the Circle, and the other meets the Circumference, in ſome Point only; and, if the Rectangle under the whole Secant and the external Segment, be equal to the Square of the other Line, that other Line is a Tangent to the Circle, and touches the Circumference in that Point.

THEOREM XVII.

If from any Point, in the Diameter of a Circle, there be drawn two Right Lines to the Circumference; one perpendicular to the Diameter, the other to the middle point of the Ark of the Semicircle; the Squares of thoſe two Lines, together, will be equal to half the Square of the Diameter.

Let AB be a Diameter, C the Center, and D the middle Point of the Semicircle, ADB.

From the Point E, draw EF, perpendicular to AB, and join ED.



I ſay, the Squares of EF and ED, together, are equal to AC Square twice taken; i.e. to half the Square of the Diameter, AB.

Join CD and CF.

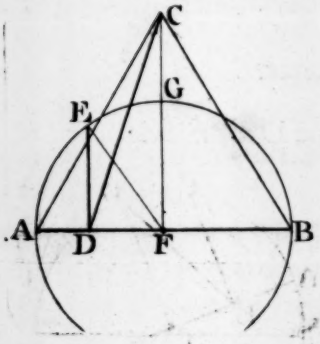
DEM. $CF\square = EF\square + EC\square$ - - - - - - 20. I
wherefore, $EF\square + EC\square = AC\square$; for $AC = CF$.
But $ED\square = CD\square + EC\square$; and $DC = CF = AC$.
Conſ. $ED\square - EC\square = EF\square + EC\square$; i.e. $= AC\square$.
wherefore $EF\square + ED\square = CF\square + CD\square$.
Therefore, $EF\square + ED\square = 2AC\square$; i.e. $= AB \times AC$.

THEO-

THEOREM XVIII.

If from the Vertex of an equilateral Triangle, conſtructed on the Diameter of a Circle, a Right Line be drawn cuting the Diameter; and, from that Point, another Line be drawn, perpendicular to the Diameter, cuting the Circumference; the Squares of thoſe two Lines, together, will be equal to the Square of the Diameter.

From C, the Vertex of the equilateral Triangle ACB, on the Diameter AB, draw CD, at pleaſure, cuting the Diameter in D; from which, draw DE perpendicular to AB.



I ſay, that the Squares of CD, DE, together, are equal to the Square of AB.

If CD be drawn perpendicular to AB, as CF, it is manifeſt.

For, CF □ + FG (equal AF) □ = AC (equal AB) □. - - - - - - 20. 1

Let CF be drawn, perpendicular to AB, and join EF.

DEM. Becauſe CF is perpendicular to AB, and AC is equal to CB, AB is biſected in F; - - C. 3. 9. 1

wherefore, F is the Center of the Circle. - Def. 20 & 21

Now, AC = AB (Con) and AC □ = CF □ + AF □ - 20. 1

But, AF = EF, and EF □ = ED □ + DF □, eq. AF □ - ſame;

wherefore, AC □ = CF □ + DF □ + DE □.

But, CD □ = CF □ + DF □ - - - - - - 20. 1.

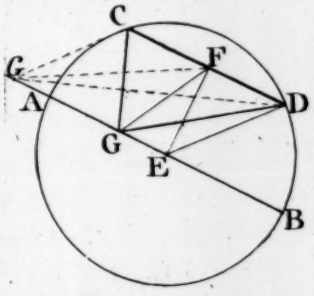
conſequently, AC □ = CD □ + DE □.

But AB = AC (Con.) Therefore, AB □ = CD □ + DE □.

THEO-

THEOREM XIX.

If a Chord Line be parallel to a Diameter of a Circle, and, from each extreme of the Chord, Right Lines are drawn to any Point in the Diameter; the Squares of thoſe Lines, together, are equal to the Squares of the Segments of the Diameter, made by that Point.



Let the Chord CD be parallel to the Diameter, AB;
and, to any Point, G, in the Diameter, draw CG and DG.

I ſay, the Squares of CG, GD, together, are equal to the Squares of AG and GB. Let E be the Center.

Biſect CD, in F; and join FG, FE and ED.

Dem. Then, becauſe CD is biſected, in F,
$CG\square + GD\square = 2FG\square + 2FD\square$ - - - - 17. 2
And, becauſe CD is biſected, EF is perpend. to CD. - 1.
But, CD is parallel to AB; conſ. EF is perpend. to AB - 4. 1
wherefore, $FG\square = EF\square + EG\square$;
and, $ED\square = EF\square + FD\square$. - - - - - - 20. 1
conſ. $CG\square + GD\square = 2GE + 2EF + 2FD$ Squares;
i. e. equal to $2GE\square + 2ED\square$. But $AE = ED$;
wherefore, $CG\square + GD\square = 2GE\square + 2AE$, equal $ED,\square$
But, $AG\square + GB\square = 2AE\square + 2GE\square$ - - - 9. 2
therefore, $CG\square + GD\square = AG\square + GB\square$. - Ax. 3. 1

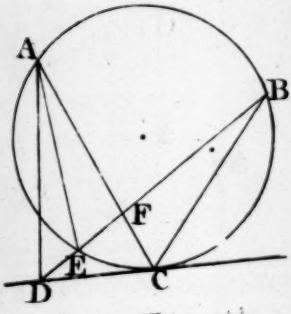
THEO-

THEOREM. XX.

If two Right Lines are drawn, from any two Points in the Circumference of a Circle, to the ſame Point in a Tangent to that Circle; thoſe Lines will make the greateſt Angle, when they meet in the Point of contact.

Let A and B be the Points aſſumed, in the Circumference of the Circle ABC.

Draw AC and BC to the Point C, in which a Tangent, DC, touches the Circle; and alſo, to any other Point, D, draw AD and BD.



I ſay, the Angle ACB is greater than ADB.

Join AE

DEM. The Angle AEB=ACB. - - Th. 10. ſtanding on the ſame Ark, AB.

But, the Angle AEB is greater than ADB for it is equal to ADB+DAE - 10. 1.

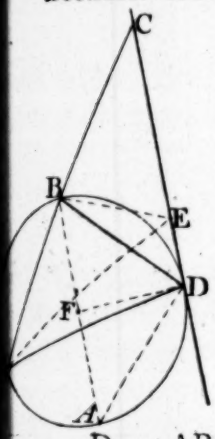
Therefore, ACB is greater than ADB.

COR. Hence it is evident, that an Angle, AFB, which falls within the Circle, is greater, and an Angle ADB without the Circle, is leſs, than any Angle, AEB, touching the Circumference, and ſtanding on the ſame, or an equal, Ark, AB.

From hence may be deduced the following Problem.

Any Right Line being given, and two Points given, or aſſumed, without the Line; to determine the Point in that Line, to which, if two Right Lines be drawn, from the given Points, they ſhall contain a greater Angle than any other Right Lines drawn, from thoſe Points, to any other Point in that Line.

A and



A and B are the aſſumed Points; and, let CD be a given Right Line.

It is required to find the Point D, ſo, that, if two Right Lines AD and BD be drawn, to that Point, the Angle ADB, ſhall be greater than any other Angle, made by Right Lines drawn, from the ſame Points A and B, to any other Point in CD.

When the given Points, A and B, are not equi-diſtant from CD.

Draw AB, and produce it till it cuts DC, in C.

Find the Point D, ſo, that the Square of CD ſhall be equal to a Rectangle under the whole Line AC and the Segment BC. (by Prob. 25, or 30.)

Then is D the Point ſought; in which, two Right Lines, AD and BD, meeting, ſhall contain a greater Angle, than to any other point in CD.

Dem. For, becauſe the Rectangle ACB is equal to the Square of CD. (Con.) if a Circle, ABD, be deſcribed, through the Points A, B, & D. (by Pr. 40.) CD will be a Tangent to that Circle, and D the Point of contact-C. 3. 16. Therefore, ADB is larger than any other Angle, AEB, made by Right Lines, drawn from the ſame Points, A and B, to any other Point, E, in the given Right Line CD.

If the given Points, A and B, are equidiſtant from the given Line; a Right Line joining thoſe Points, being biſected, and a Perpendicular, to that Line, drawn, from the Point of biſection, cuting the given Line, will give the Point required.

For, a Right Line joining the Points will be parallel to the given Line, (Def. 7.) wherefore, a Line perpendicular to it is alſo perpendicular to the given Line (3. 1.) and will, conſequently, cut it in the Point of contact of a Circle deſcribed through that Point and the two given Points (C. 1. 1. and 2. 8. 3.)

ELEMENTS OF GEOMETRY.

BOOK IV.

THE fourth Book of Euclid's Elements is not properly elementary, but practical or mechanical; it treats of the inſcription and circumſcription of right-lined Figures in and about Circles. It is of uſe in Trigonometry, Aſtronomy, &c. and alſo, in Fortification or military Architecture, which ſeems to depend entirely on it.

As I do not think it proper to deviate from the order of Euclid, in his Books, I have therefore made it the fourth; otherwiſe, I ſhould have given it a place amongſt the other Problems, in practical Geometry, it being entirely problematical. I have given Demonſtration of each Propoſition, as brief as it will poſſibly admit of; becauſe, I would not have the Work deficient in any part; which would be ſufficient reaſon, with ſome Perſons, to condem the whole. Nevertheleſs I am of Opinion, that great part of it does not require Demonſtration; being ſufficiently evident from the Conſtruction; eſpecially, as no other part of the Elements is dependant on it.

There are, in ſome Books extant, ſundry other Propoſitions relative to the inſcription and circumſcription of right-lined Figures, in and about other right-lined Figures; particularly in Stone's Euclid; and in a curious Tract of practical Geometry by Le Clerc. But, as there is nothing of real utility in them, nor any thing extraordinary to recommend them

them to the Curious, I ſhall not, by uſeleſs additions, to this, detain the Reader from matter of greater Importance in the fifth and ſixth Books. Indeed this may be paſſed over entirely, for the preſent, and proceed immediately to the fifth; there being nothing, in the fourth, neceſſary to be known previous thereto.

DEFINITIONS.

I have already, in the General Introduction, defined the Terms, to deſcribe, to inſcribe, and to circumſcribe; a repetition of which would be unneceſſary.

1. A Right-lined Figure is ſaid to be inſcribed, in a Circle, or to have a Circle circumſcribing it, when every Angle of the Figure, touches the Circumference of the circumſcribing Circle.

2. A Right-lined Figure is, then, ſaid to circumſcribe a Circle, or to have a Circle inſcribed, when every Side of the Figure touches the Circumference of the Circle.

3. A Right-lined Figure is ſaid to be inſcribed, or to circumſcribe a Right-lined Figure; when every Angle of the inſcribed one, touches every Side of the circumſcribing Figure.

4. So likewiſe, A Circle is ſaid to be inſcribed, in a Right-lined Figure, when it touches every Side of the Figure; or to circumſcribe a Right-lined Figure, when the Circumference touches every Angle.

N. B. By inſcribing any Figure within a Circle, or any Right-lined Figure within another, is underſtood, the deſcribing a Figure, like or ſimilar to the given one, the largeſt poſſible, to be contained in the other.

5. A Right Line is ſaid to be applied to a Circle, when each extreme is in the Circumference.

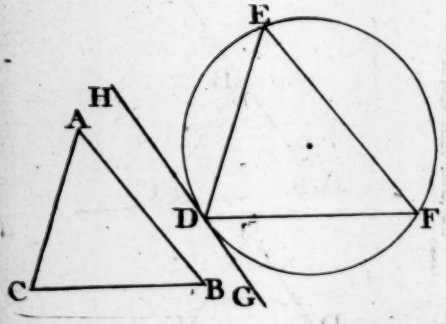
N. B. To inſcribe a Right Line in a Circle, is the firſt Problem of Euclid's Fourth Book, which I think entirely unneceſſary, as a Problem, nothing more being required, in the Operation, than to take the given Line, in the Compaſſes, and placing one point in the Circumference, at pleaſure, to cut the Circumference, with the other.

A Right Line, greater than a Diameter, cannot be inſcribed in a Circle.

All the Propoſitions of this Book, follow in the order of Euclid, omitting the Firſt.

PROPOSITION II.

To inſcribe a Triangle in a given Circle, equiangular to a given Triangle.



Let ABC be the given Triangle, to be inſcribed in the Circle DEF.

Draw at pleaſure the right line GH, touching the Circle in any point of its Circumference; as D.

Make the Angle GDF equal to A. i. e. to any Angle of the Triangle; and HDE equal to another, B, and join EF.

DEM. The Angle GDF (equal A) is equal to DEF; and HDE (equal B) = DFE - - - - - - 13. 3

Wherefore, the Angle DEF, being eq. A, and DFE eq. B, the remaining Angle, EDF, muſt neceſſarily be equal to C.

Th. the Triangle DEF is equiangular to ABC. - 10. 1

N. B. By taking the Angles in this order, the Triangle inſcribed will be poſited as the given one; but any two being taken, the Triangle, inſcribed, will be the very ſame.

Otherwiſe:

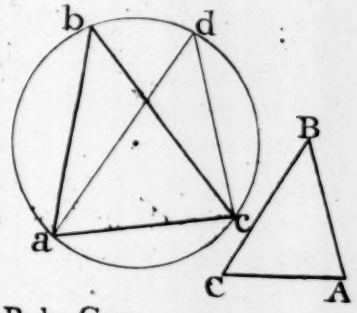
Otherwiſe:

Draw at pleaſure a d, cuting the Circumference in a and d.

Make the Angle, a d c, equal to B, & join a c.

Make, c a b, equal to the Angle A, & join b c.

The Triangle a b c is equiangular to the Triangle ABC.



For, the Angle a b c = a d c (10. 3.) equal B, by Con.
And, the Angle c a b was made equal to A;
Conſ. a c b is equal to the remaining Angle C. - 10. 1
Therefore, the Triangle, a b c, is equiangular to ABC.

PROPOSITION III.

To circumſcribe, that is, to deſcribe or draw, a Triangle, about a given Circle (touching it on every Side) equiangular to a given one.

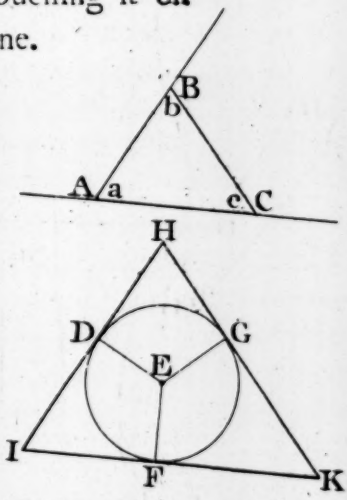
Let ABC be the given Triangle, and DFG the Circle given.

Produce any Side of the Triangle, as AC, both ways; and draw a Radius, DE.

Make the Angle DEF, equal to the external Angle A; and FEG equal C.

Through the Points D, F, and G; draw Tangents to the Circle, cuting each other in H, I and K.

Then is H I K the Triangle required.



N.B. Two Angles, DEF, DEG, or FEG, being made equal to any two external Angles of the Triangle, will give the ſame Triangle, H I K, but differently poſited.

DEM.

DEM. Becauſe the Right Lines, HI, IK, and HK, touches the Circle, in the Points D, F, and G; and, from thoſe Points, the Lines DE, FE, and GE are drawn to the Center, the Angles, EDI, IFE, &c. are right ones. - 8. 3

And, becauſe the Angles, D, E, F, and I, of every Quadrilateral, are equal to four Right Angles, - 11. 3

the Angle I + DEF = two Right ones.

But, the Angle DEF is equal to A, by Conſtruction.

conſequently, D I F is equal to BAC. - - 1. 1. & Ax. 7.

After the ſame Manner, the Angle K may be proved equal to ACB; and the Angle H, equal ABC.

Therefore, the Triangle HIK is equiangular to A B C.

The Reaſon of this appears obvious, from the two Theorems deduced from the 10th Propoſition, Book I. For, ſince (by the firſt) all the Angles of every Right-lined Figure are, together, equal to twice as many Right Angles as it has Sides, wanting four; and by the 2nd, all the external Angles are equal to four Right Angles. Alſo, by the 10th, the three Angles of every Triangle are equal to two Right Angles; and all the Angles, about a Point, are equal to four; (2.1.) conſequently, ſince the three Angles DEF, DEG, and FEG, are reſpectively equal to the external Angles, A, B, and C, of the Triangle ABC; the Angles H, I, and K are, alſo, reſpectively equal to the internal Angles, a, b, and c, of the Triangle.

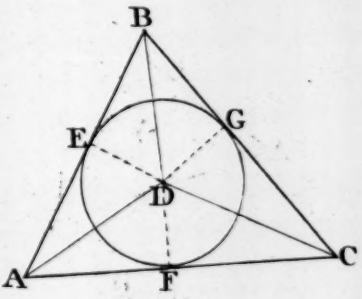
PRO-

PROPOSITION IV.

To inſcribe a Circle in a given Triangle, (ABC) touching every Side of the Triangle.

Biſect any two Angles of the Triangle, ABC, and CAB, by the Right Lines BD, and AD, cuting each other in D; from which Point, draw a Perpendicular (DE) to any Side (AB) of the Triangle.

With the radius DE, on the Center D, deſcribe the Circle EFG, which will touch every Side of the Triangle, ABC.



Draw DF & DG, perpendicular to the Sides, AC & BC.

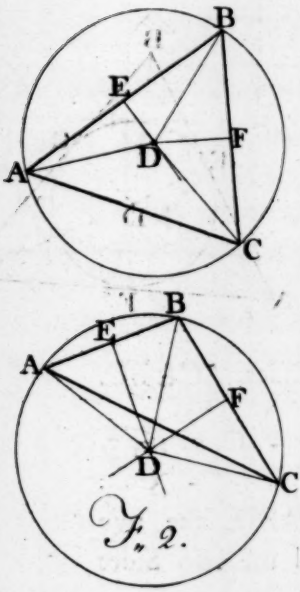
Dem. Now, AF is equal to AE - - - - C. 2. 16. 3.
and, the Angle EAD is equal to DAF. - - - Con.
Wherefore, in the Triangles AED, AFD, the two Sides AE, AD, are reſpectively equal to the two Sides AF, AD; and they contain equal Angles.
Therefore DE is equal to DF. - - - - - 8. 1

After the ſame manner, DG may he proved equal to DE, and alſo to DF.

Wherefore, the three Lines DE, DF, and DG, being perpendicular to the three Sides of the Triangle, and being proved equal among themſelves, the Circle, EFG, will paſs through the three point E, F, and G; and, conſequently, will touch every Side of the Triangle ABC.

PROPOSITION V.

To circumſcribe a given Triangle with a Circle; or, to deſcribe a Circle about a given Triangle.



ABC is the Triangle given.

Biſect any two Sides of the Triangle, AB and BC, in E and F.

Draw ED and FD, perpendicular to AB and BC, cuting each other in D, the Center of the circumſcribing Circle.

Fig. 2.

If the Triangle has a Right Angle, the Hypothenuſe biſected gives the Center.

If an Angle be obtuſe, the Center of a circumſcribing Circle falls without the Triangle; the Operation is the ſame in both, (ſee Fig. 2.)

Join the Points AD, BD and CD.

DEM. Becauſe the Side AB is beſected in E, and DE is perpend. to AB, the Sides AE, ED, of the Triangle AED, are equal, reſpectively, to BE, ED. of the Triangle BED; and the Angle AED is equal to BED. - - Def. 10. wherefore, AD is equal to BD - - - - - - 9. 1. for they ſubtend equal Angles, in the Triangle ABD.

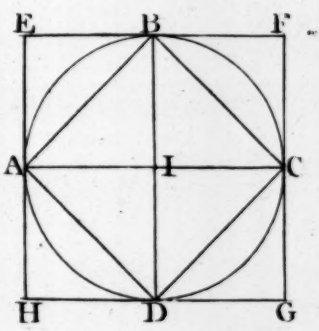
In the ſame manner may be proved, CD=AD, or BD. Conſequently, D is the Center of the circumſcribing Circle, ABC.

PRO-

PROPOSITION VI.

To inſcribe a Square in a given Circle; and to deſcribe a Square about a Circle. ABCD and EFGH.

Draw two Diameters, AC and BD, at Right Angles. Join the Extremes, A, B, C, D, and a Square is inſcribed.



2nd. Draw Tangents through the extreme Points of the Diameters, meeting in E, F, G, and H; or draw Lines, touching the Circle, parallel to each Side of the inſcribed Square, and EFGH is a Square circumſcribing the Circle.

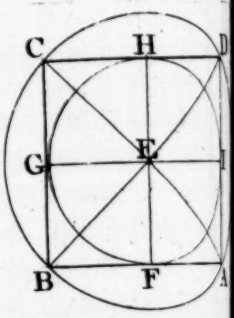
Dem. The Diameters AC & BD biſect each other, - 16. 1
therefore, AI, IC, IB, and ID, are equal;
and they contain equal Angels, AIB=BIC &c. - - Con.
Wh. AB, BC, CD & AD, are all equal amongſt themſelves.
And, the Angles ABC, BCD, &c being in a Semicircle, are Right Angles. Therefore ABCD is a Square.

2ndly. Becauſe EF touches the Circle, in the Point B, it is perpendicular to the Diameter BD. - - - - 8. 3
And, becauſe HG is perpend. to BD, HG is parallel to EF;
For the ſame reaſon, EH is parallel to FG;
Therefore, EFGH is a Parallelogram. - - - Def. 33.
But, AC is perpendicular to BD. (Con.) wherefore, EH is perpendicular to HG, and ſo is EF to FG and to EH; conſequently, EFGH is a Rectangle. - - - Def. 34
But EF, FG, EH, and HG, are each equal to AC, equal BD.
Th. EFGH is equilateral; conſ. it is a Square - Def 35

N. B. The Square, EFGH, circumſcribing a Circle, is double the Square, ABCD, inſcribed in the ſame Circle.
For the Triangle ABC = half the Rectangle AEFB - - 17. 1
And ADC is equal to half the Rectangle AG;
conſequently, the Square EFGH is double ABCD.

PROPOSITION VII.

To inſcribe a Circle in a Square; and to circumſcribe i. e. to deſcribe a Circle about a Square.



Let ABCD be a given Square

Draw the Diameters AC & BD cuting each other in E.

From the Center E, draw EF perp. to AB.

With the Radius EF, on the Center E, deſcribe a Circle, which will touch every Side of the Square ABCD.

Draw EG perpend to BC; produce GE to I, and FE to H.

DEM. Now, becauſe ABCD is a Parallelogram, the Diagonals AC & BD biſect each other, in E; - - - - 16. 1

And becauſe AB, BC, are reſpectively equal to AB, AD, and contain equal Angles; AC=BD - - - - - 8. 1

conſequently their halves are equal. - - - Ax. 1. 1

Then, ſince AE, EB, and EC are equal, the Triangles AEB, BEC are Iſoſceles; wh. AB and BC are biſected by the Perpendiculars EF & EG, conſ. FB=BG. - C. 4. 9. 1

But, ABC is a Right Angle, and AB=BC. - Def. 35

and ſince AC is biſected in E, EB is perpendicular to AC

Wh. AEB being a Right Angle ABE, EAB, are half Right; conſequently, BE biſects the Right Angle ABC.

Now ſince in the Triangles EFB, BGE, the Sides, FB, BE, are reſpectively equal to GB, BE; and the Angle FBE=EBG, the remaining Side EF=EG.

After the ſame manner EH and EI may be proved equal to EF and FG; conſ. the Circle FGH touches every Side of the Square.

2ndly.

2ndly. Having drawn the Diameters of the Square AC & BD; with the Radius EA or EB, deſcribe the Circle, ABCD.

This needs no further Demonſtration.

For, ſince the Diagonals of a Square are equal, and mutually biſect each other, (16. 1.) the halves EA, EB, EC, and ED are alſo equal; wherefore, a Circle, ABC, will paſs through every Angle of the Square.

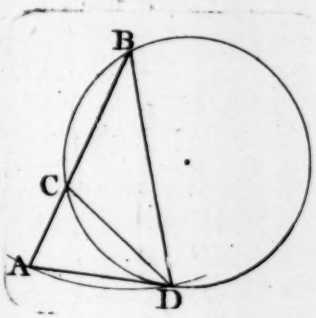
N. B. A Circle (FGHI) inſcribed in a Square, is equal to half a Circle (ABCD) circumſcribing that Square.

For, the Square of AC = AB □ + BC □ ; conſequently, AC □ is double BC, equal FH, □.

But the Areas of Circles are, to each other, as the Squares of their Diameters, - - - - - - - - - - C. 1. 14. 6. therefore, the Circle ABCD is double FGHI.

PROPOSITION VIII.

To make an Iſoſceles Triangle, having its Angles, at the Baſe, each double the Angle at the Vertex.



Let AB be any given Right Line for one of the equal Sides.

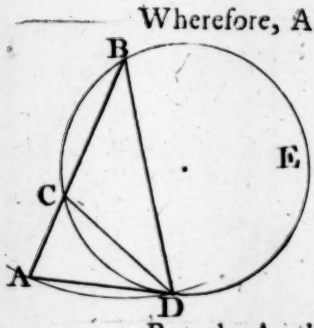
Divide AB in extreme and mean Proportion, in C. (Prob. 35.)

With the Radius AB, on B, deſcribe an Ark, AD.

Make AD equal to CB, and draw BD.

I ſay; in the Triangle ABD, the Angles A and D, at the Baſe, are each double the Angle B, at the Vertex.

Join C D, and deſcribe a Circle through B, C, and D, (Prob. 40.) or, by the 5th of this.



DEM. Becauſe the Rectangle BAC=BC□ - - - 11. 2.
And AD=BC (Con.) the Rectangle BAC=AD□;
Wherefore, AD touches the Circle BDE, at D - 16. 3
(For A is a Point without, that Circle, from which, two Right Lines are drawn; one (AB) cuting, the other, (AD) touches the Circle;)
Therefore, the Angle ABD=ADC. - 13. 3
(For, CD cuts the Circle BDE, in the Point of contact, D.)
But, the Angle ACD=ABD+CDB, i.e.=ADC+CDB;
And, the Angle BAD=ADB (9. 1) for AB=BD. - Con.
conſequently, BAD=ACD; wh. CD=AD - C. 3. 9. 1
But, AD=CB (Con.) wherefore, CD=CB; - Ax. 3. 1
conſequently, the Angle CDB=CBD - - Th. 9. 1
But, the Angle ADC=CBD; (proved, above)
wherefore, ADC is equal to CDB. - - - Ax. 3. 1
and therefore, ADB, BAD are each double ABD.

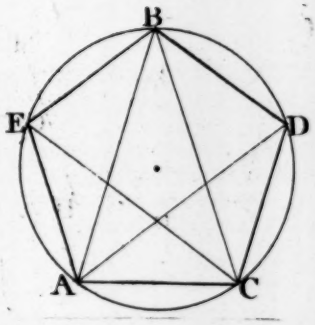
The Conſtruction of this particular ſpecies of Triangles, in reſpect of itſelf, is of little or no uſe; or, I ſhould have given it a place, amongſt the other Problems, in Practical Geometry. But, the inſcribing a regular Pentagon in a Circle, according to Euclid, in the next Propoſition, depends on it entirely; but, which may, for real uſe, in practice, be more readily conſtructed. Nevertheleſs, for the manner and elegance of its Demonſtration, I did not think proper to omit it.

PRO-

PROPOSITION IX.

To inſcribe a regular Pentagon in a Circle.

A B C is the given Circle.



Inſcribe an Iſoſceles Triangle, ABC, whoſe Angles BAC, ACB are, each, double the Angle ABC - - Prop. 1
Biſect each of the Angles BAC, ACB by the Right Lines AD and CE, cuting the Circumference in D and E.
Join the Points, A & E &c. by the Right Lines AE, EB, BD, and DC.
AEBDC is a regular Pentagon.

Dem. Now, becauſe each Angle, BAC, ACB, is double the Angle ABC; (Con.) and thoſe Angles are biſected, the Angles, ABC, ACE, ECB, BAD, and DAC, are equal to one another.

But equal Angles ſtand on equal Arks, - - C. 2. 9. 3
and, equal Arks have equal Chords or Subtenſes, - 2. 3. 3
wherefore, the Right Lines AC, AE, EB, &c. are all equal; and conſequently, the Pentagon, A E B D C, is equilateral.

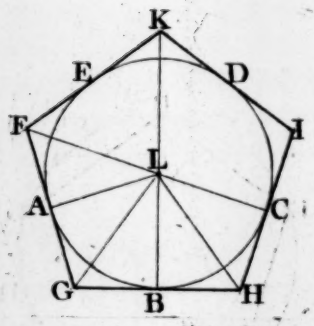
I ſay, it is alſo equiangular.

For, becauſe the Arks AE, EB, &c. are all equal, the Angles ACE, ECB, &c. are equal.
wh. the Angle EBA + ABC + CBD = EAB + BAD + DAC.
conſequently the Angle EBD = EAC; and ſo, of all the reſt.

Or, becauſe each Angle of the Pentagon, AEB, EBD, &c. ſtands on equal Arks, ACDB, EACD, &c; (for each is triple the Ark, AC.)
Therefore, the Pentagon A E B D C is equiangular.

PROPOSITION X.

To deſcribe a regular Pentagon about a Circle.



Let ABCDE be the angular Points of a Pentagon, inſcribed in a Circle, through which draw the Tangents FG, GH, HI, &c. cuting each other, in F, G, H, I, and K, i. e. having, made the Arks AB, BC, &c. equal, (viz. each equal 72 Degrees) through the Points A, B, C, &c. draw the Tangents, as before.

The Pentagon FGHIK is equilateral and equiangular.

Let L be the Center of the Circle; draw AL, BL, &c. alſo draw FL, GL, &c.

DEM. AL, BL, &c. are perpendicular to FG, GH, &c. - 8 3
and they are all equal between themſelves - - - Ax. 1. 3
The Angle ALB is equal to BLC, by Conſtruction.
AG is equal to GB, and BH equal to HC - - C. 2. 16. 3
wh. the Triangles ALG, GLB, BLH, &c. are equilateral and equiangular to each other; for GB=BH, &c. - 11. 1
AF=AG, and CI=CH, (ſame) conſ. FG=GH, - Ax. 1
After the ſame manner, FK and IK, may be proved equal between themſelves, and alſo to FG, &c;
therefore, FGHIK is equilateral.

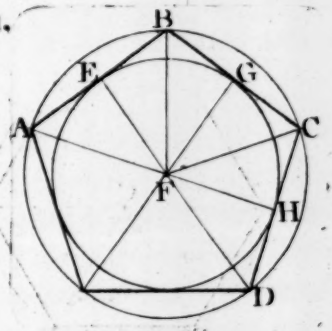
And, ſince the Triangles ALG, GLB, &c. are equiangular, the Pentagon is alſo equiangular.
For, the Angles AGL, LGB, alſo BHL and LHC are equal amongſt themſelves; conſequently, FGH=GHI.

After the ſame manner, the Angles, I and K, may alſo be proved equal betwen themſelves, and alſo to F, G, & H.
Th. FGHIK is a regular Pentagon, circumſcribing a Circle.

Note. If the Circumference of a Circle be divided in five equal Parts, and the Points joined by Right Lines; then if Tangents to the Circle be drawn, parallel to every Chord, a regular Pentagon will be circumſcribed.

PROPOSITION XI.

To inſcribe a Circle in a regular Pentagon; and, to deſcribe a Circle about a Pentagon.



Biſect any two Sides, AB, and BC, of the Pentagon ABCD, by the Perpendiculars EF and FG, interſecting in F; on which, with the Radius EF or FG, deſcribe a Circle; which, will touch every Side of the Pentagon.

2nd. With the Radius AF or FB (the Point F being found as before, or by biſecting two Angles, ABC and BCD) deſcribe a Circle, which will paſs through every Angle of the Pentagon.

Dem. Becauſe AB = BC, EB = BG. Draw AF, FB, &c. Then, in the Triangles EBF, FBG; FB□ = FE□ + EB□. and, FB□ = FG□ + BG□; - - - - - 20. 1. conſequently, FE□ + EB□ = FG□ + BG□. - Ax. 3. 1 But, EB = BG (Ax. 1. 1.) wherefore, EB□ = BG□; conſequently, EF□ = FG□; and, therefore EF = FG.

In the ſame manner, FH may be proved equal to FG; F is, therefore, the Center of a Circle inſcribed.

2nd. Becauſe AB is biſected, in E, AE = EB, & AE□ = EB□ Add EF□, to both; and AE□ + EF□ = EB□ + EF□. But, EF is perpendicular to AB, by Conſtruction; conſequently, AEF, FEB are Right Angles. - Def. 10 wh. AF□ = FB□; and FB□ = EB□ + EF□. Conſequently, AF□ = FB□; and therefore, AF = FB.

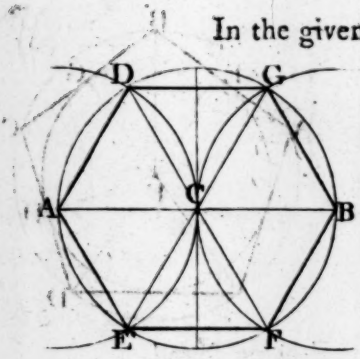
In the ſame manner, FC, FD, &c. may be proved equal to AF and FB.

conſequently, F is the Center of a circumſcribing Circle.

PRO-

PROPOSITION XII.

To inſcribe a regular Hexagon in a given Circle.



In the given Circle, AGF, draw a Diameter, AB.

With the Radius of the Circle, AC, on the Centers A and B, deſcribe two Arks, DCE, FCG, cuting the Circumference in the Points D and E, F and G.

Or, having drawn the Ark DCE, only, cuting the Circle in D and E, draw DC & EC. and produce them to F and G.

Join the Points A and D, AE, EF, &c. by Right Lines, which compleats the Hexagon, ADGBFE. Q. E. F.

I ſay, it is both equilateral and equiangular.

DEM. Becauſe C is the Center of the Circle AGF, the Lines DC, AC, and CE are equal; and, becauſe A is the Center of the Circle DCE, AD, AC and AE, are equal; wherefore, ADC & ACE, are equal & equilateral Triangles.

But, the Angles of an equilateral Triangle are, each, equal to one third Part of two Right Angles. - C. 10. 1 (becauſe, the Angles which are oppoſite to equal Sides are equal; and the three Angles of every Triangle are, together, equal to two Right Angles. - - 9 and 10. 1) conſequently, DCA+ACE= two thirds of two R. Angles.

Now, becauſe the Right Line EC falls on the R. Line DF, the Angle ECF+ECD= two Right Angles - 1. 1 and, becauſe DCA+ACE (eq. DCE)=two thirds of two Right Angles; ECF is equal one third part of two Right Angles; i. e. equal to ACE, equal ACD.

Again

Again; becauſe the Right Lines AB, DF, & EG cut one another, in the Point C, the Angles DCG, GCB, & BCF are, reſpectively, equal to ECF, ACE, and ACD - 2. 1 Wh. thoſe ſix Angles are all equal amongſt themſelves; conſ. they ſtand on equal Arks; and conſ. on equal Chords. Therefore, the Sides, AD, DG, GB, &c. of the Hexagon, are equal to one another.

2nd. Becauſe the Triangles, ADC, CDG, &c. are equilateral, and conſequently equiangular, - - - - - C. 2. 9. 1. the Angle ADC + CDG = DGC + CGB, - - Ax. 3. i. e. the Angle ADG = DGB; and ſo of all the reſt. Therefore, the Hexagon ADGBFE, inſcribed in a Circle, is equiangular; and alſo equilateral.

Cor. The Side of a regular Hexagon, inſcribed in a Circle, is equal to the Radius, or Semidiameter of the Circle.

For, the Triangles, ADC, &c. are proved to be equilateral.

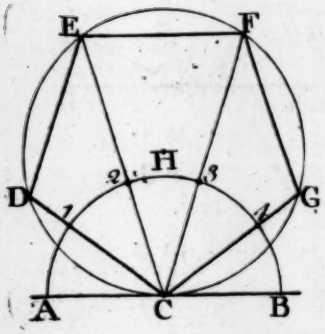
Hence, a Duodecagon may be inſcribed in a Circle, by biſecting each Ark on the Side of a Hexagon inſcribed. i. e. The Radius of the Circle being applied ſix times in the Circumference, each Ark biſected gives the Sides of a Decagon.

N. B. Any Poligon may be circumſcribed about a Circle, having firſt inſcribed one (of the Species required) by drawing Tangents, through every Angle, or parallel to every Side, of the inſcribed Figure.

In Prob. 52, Practical Geometry, is given a general method for inſcribing Poligons of every Denomination in Circles; of which Method, there is no Demonſtration extant, having conſulted ſome able Mathematicians concerning it.

I ſhall, here, give another general method, which admits of Demonſtration (granting an Ark of a Circle to be divided into any number of equal Parts) which is as follows.

Let DFC be a given Circle, in which it is required to deſcribe a regular Pentagon.



Draw AB, at pleaſure, touching the Circle DFC in any Point, C; on which Center, draw a Semicircle AHB, with any Radius at diſcretion.

Divide the Semi-circumference into five equal Parts, i. e. make the Arks A 1, 1 2, 2 3, &c. each equal 36 Degrees.

Draw C 1, C 2, and produce them till they cut the Circumference, in D, E, F, and G.

Join DE, EF and FG, and it is done.

DEM. Becauſe the Angs DCE, ECF, &FCG are equal - Con. the Arks, and conſ. the Chords DE, EF, &FG, are equal, Alſo, the Angle ACD (eq. DCE by Con.)=DEC. - 13.3 wh. the Triangle CDE is Iſoſceles; and CD=DE. - 5.1

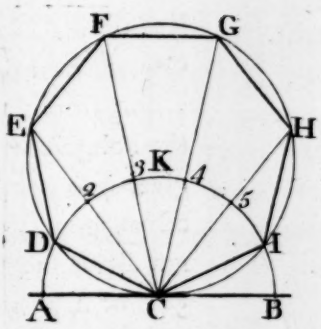
After the ſame manner CG may be proved equal to FG. conſequently DC, DE, EF, FG, and GC are all equal. Therefore, the Pentagon DEFGC is equilateral; and, being inſcribed in a Circle, it is neceſſarily equiangular, each Angle, CDE, DEF, &c. ſtanding on equal Parts of the Circumference, CGFE=DCGF.

This method of inſcribing Poligons is not in Euclid, nor in any Author I am acquainted with. It is neither ſo operoſe in the Conſtruction, and is more briefly demonſtrated.

Euclids method (for a Pentagon) of inſcribing a Triangle ſimilar to another, (which muſt firſt be formed) whoſe Angles at the Baſe are double of the Vertex, is very ingenious and perfectly Geometrical; but it is liable to great error in the Conſtruction; inſomuch, that it is extremely difficult to do it, with accuracy, and tedious in the Operation; whereas, this is both more expeditious and more to be depended on.

After

After the ſame manner, a Heptagon, or any Poligon whatever, may be readily conſtructed, as follows.



Draw AB, at pleaſure, touching a Circle in C; on which Center deſcribe a Semicircle, cuting the Tangent in A and B.

Divide the Ark of the Semicircle into as many equal Parts, as the Poligon, required, has Sides, viz. Seven (for a Heptagon) and, through every Diviſion, draw C1, C2, &c. cuting the given Circle in D, E, F, &c; which will divide the Circumference into ſeven equal Parts, in thoſe Points; and, being joined by Right Lines, compleats the Heptagon CDEFGHI.

The Demonſtration is the ſame as in the preceding Propoſition. But, it may be further obſerved, that, becauſe the equal Angles ACD, DCE, ECF, &c. are at the Center of the Circle AKB, and at the Circumference of the Circle EHC; and, becauſe the Angle at the Center of a Circle, on any Ark, is equal to an Angle at the Circumference, on twice that Ark (C. 1. 9. 3.) therefore, the Semi-circumference, of one and the whole Circumference, of the other, are divided into the ſame number of equal Parts, by the Right Lines, CD, CE, &c.

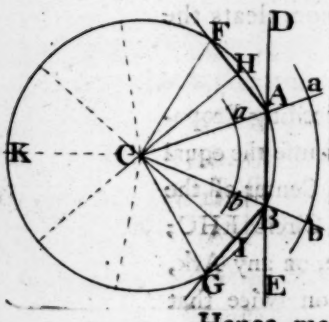
If AC be taken equal to the Radius of the given Circle, it is manifeſt.

It may be alledged, that the whole Circumference may as eaſily be divided into any number of equal Parts, as a Semi-circumference. 'Tis true it may, but that admits of no other Demonſtration, the Sides being all equal by Conſtruction; and is in no wiſe geometrical.

On account of the ſingularity of this method, having never ſeen it in any other work, and becauſe I find it uſeful in Perſpective, in ſome Caſes, I thought proper to inſert it.

The Side of any Poligon inſcribed in a Circle of any Radius is readily obtained, by making an Angle at the Center, of the number of Degrees, which is the Product, ariſing from the Diviſion of 360 by the Denomination of the Poligon. e. g. 360 divided by 5 (for a Pentagon) gives 72; for a Hexagon, 60; for an Octagon, 45; for a Nonagon, 40; for a Decagon, 36. A Heptagon does not divide the Number of Degrees exactly; but is nearly, 51⅓, or more nearly, 51, 43.

Wherefore, having (by means of a Protractor, or Line of Chords) made an Angle, ACB, at the Center of a Circle, equal 40 Degrees; the Chord AB, of the Ark it ſtands on, is the Side of a Nonagon inſcribed in the Circle FKG; i. e. AB is one ninth Part of the whole Circumference.



For, let the Angle ACB be conſtructed or meaſured on any Ark whatever, as ab, either of a greater or leſs Radius, the portion AB, of the Circle AKB, will be the ſame; for AB and ab are each one ninth part of their reſpective Circumferences.

Hence may be deduced, an univerſal method for Conſtructing Poligons of any number of Sides, on a given Right Line, viz. by finding, what Angle the Side of the Poligon, ſubtends at the Center, as above; which in a Nonagon is 40 Deg. Subtract 40, from 180, two Right Angles; the remainder is 140, half of which is 70.

Conſtruct an Iſoſceles Triangle, ACB, on the given Line AB, whoſe Angles, at the Baſe, A and B, are each of 70 Degrees; the Angle at the Vertex, C, will be 40.

C is the Center of a Circle which will circumſcribe a Nonagon, the meaſure of whoſe Sides is AB.

With

With the Radius CA or CB deſcribe a Circle, AKB, it will contain AB nine times, applied to the Circumference. For, if the Angle ACB, of 40 Degrees, be repeated 9 times, as in the Figure; it will include all the ſpace of the Circle, equal four Right Angles; and each Angle will be ſubtended by a Side of the Poligon; which is evident from the Figure.

The three Angles of every Triangle are, together, equal to two Right Angles, i. e. equal to 180 Degrees. (10. 1.) And, ſince the Angle C, at the Center of the Triangle ACB, is equal to 40 Degrees, conſequently, the two remaining Angles, CAB, CBA, at the Baſe, being equal (9. 1.) they are, each, equal to half the difference betweet. the Angle ACB, and two Right Angles; i. e. of 70 Degrees, each, as by Conſtruction.

Otherwiſe. The external Angle of every regular Polygon is equal to an Angle at the Center, ſubtended by a Side.

For, all the external Angles of every Right-lined Figure are equal to four Right Angles (Th. 2. 10. 1.) and ſo are all the Angles about the Center of a Circle. - - - - C. 2. 2. 1

Wherefore, an ordinate or regular Poligon, having all its internal Angles equal, will, likewiſe, have all its external Angles equal; and, being equal in number to the Angles at the Center, ſubtended by the Sides, each external Angle is, conſequently, equal to an Angle at the Center, ſubtended by a Side, e. g.

If the Poligon has eight Sides (an Octagon) the external Angle is the 8th part of 360, i. e. 45 Degrees; if a Nonagon, it is 40 Degrees; if a Decagon 36; each being equal to the Angle at the Center, ſubtended by a Side. Hence.

Let, AB be a Side given for a Nonagon; let it be produced both ways, towards D and E.

At the extremes, A and B, make the Angles DAF, EBG, each equal 40 Deg. and make AF and BG each equal AB.

Biſect

Biſect AF and BG, in H and I, and draw HC and IC perpendicular to AF and BG, interſecting in C; which, is the Center of a circumſcribing Circle, as before.

It is evident, that the external Angle DAF (equal EBG) is equal to the Angle ACB at the Center.

For, ACB, BCG, and ACF, are congruous Iſoſceles Triangles; whoſe Angles at the Baſe, AB and BG, are 70 Degrees each; two, of which, form an internal Angle of the Nonagon; as ABC + CBG = ABG; and, the external Angle, EBG, is the Compliment of two Right Angles; conſequently, it is equal to the remaining Angle ACB, of the Triangle ACB, or BCG.

Hence it is alſo evident; that, if any Right Line, as DE, cuts a Circle, and if from either Point, A or B, of the part AB, within the Circle, another Chord, AF or BG, be drawn, equal AB; the external Angle, DAF or EBG, made by that Chord and the Line DE, is equal to the Angle at the Center of the Circle, ſubtended by the Chord, AF or BG (equal AB).

From what I have advanced, is deduced the Table, after Prob. 47, Practical Geometry, for conſtructing Poligons; by dividing the Ark of a Quadrant, or Right Angle, into as many equal parts as the Poligon has Sides.

For, the Ark of a Right Angle being divided into five equal parts, each will be 18 Deg. wherefore, the Angle of a Pentagon being ſix of thoſe parts, or one added to a Right Angle, i. e. 18 + 90 = 108; which being ſubtracted from two Right Angles, or 180 Degrees, the Difference is 72, equal four times 18; which is the Compliment of two Right Angles; conſequently, the external Angle of a Pentagon is equal to an Angle at the Center, ſubtended by a Side.

For a Nonagon, the Ark of a Right Angle is divided into 9 equal parts, each equal 10 Degrees.

Now, the external Angle being 40 Degrees, equal to the Angle at the Center, the Angle of the Nonagon is 50+90 or five

five ninth parts added to a Right Angle; for, an internal and external Angle are, together, equal to two Right Angles - 1. 1

Therefore a Right Angle is to the Angle of a Nonagon as 9 to 14, difference 5, as in the Table.

The reaſon of all this is ſo very obvious, it is almoſt needleſs to ſay more about it; ſeeing it is manifeſt, that all the external Angles of any Right-lined Figure, whatever, are, together, equal to all the Angles about a Point, ſubtended by the Sides. i. e. equal to four Right Angles. And, all the internal Angles of any Right-lined Figure are equal to twice as many Right Angles as the Figure has Sides, wanting four, (Th. 1. 10. 1.) conſequently, the external Angles being equal to thoſe four (Th. 2. of the ſame) are equal to all the Angles at the Center; and, being equal, in number, they are equal in quantity.

Hence, the Angle of any regular Poligon, whatever, may be readily obtained.

For, in a Pentagon, all the internal Angles, together, are equal to ſix Right Angles; conſequently, each Angle is 6 fifths of a Right Angle, i. e. it is equal to one Right Angle and one fifth part of a Right Angle; ſeeing, there are five Angles in the Figure, and they are, altogether, equal to ſix Right Angles.

The Angles of a Hexagon are, altogether, equal to eight Right Angles; conſequently, each Angle is equal to eight ſixths of a Right Angle, or four thirds; i. e. equal to one Right Angle and one third part of a Right Angle.

A Heptagon has all its Angles, together, equal to ten Right Angles; wherefore, each Angle is equal to ten ſevenths of a Right Angle, i. e. equal to a Right Angle and three ſeventh parts of a Right Angle.

All the Angles of an Octagon being equal to twelve Right Angles, it is evident that each Angle is equal to one Right Angle and a half; i. e. to 12 eights, i. e. to ſix fourths, or three ſeconds, i. e. one and a half.

To particularize more would be unneceſſary; as it is eaſy, from what I have ſaid, to calculate the quantity of the Angle of any Poligon whatever, by the Proportion it bears to a Right Angle, or to two Right Angles; which muſt ever be more than the Angle of any Polygon whatever.

For Poligons which have an even number of Sides, it may be obſerved, that the diviſion of a Right Angle, may be reduced to a lower Denomination; as for a Duodecagon, for inſtance; all its Angles, together, being equal to 20 twelfths, i. e. 10 ſixths, or 5 third parts, i. e. to one Right Angle and two thirds; the Right Angle need be divided into three equal Parts, only, inſtead of twelve.

Whereas, thoſe which have an odd number of Sides, cannot be reduced lower; but, in forming them, a Right Angle muſt neceſſarily be divided into as many parts as the Figure has Sides.

The method of inſcribing a Quindecagon in a Circle, according to Euclid, is a matter of mere curioſity; firſt, to inſcribe an equilateral Triangle, and afterward a Pentagon, in the ſame Circle, in order to get a fifteenth part of the Circumference; which, notwithſtanding it admits of perfect Demonſtration, is liable to great error in Practice.

PROPOSITION XIII.

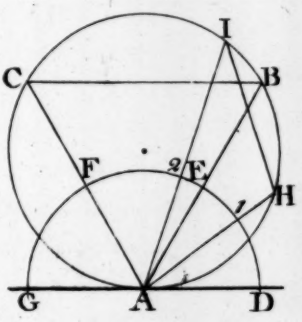
To find the Side of a Quindecagon inſcribed in a Circle.

Let ABC be a given Circle, in which it is required to inſcribe a Quindecagon.

Draw, DG, at pleaſure, touching the Circle ABC in A. With any Radius, AD, on A, the Point of contact, deſcribe a Seme-circle, DEFG.

Inſcribe the equilateral Triangle ABC; by making the Arks DE, EF and FG, each equal to the Radius AD, and drawing AE, AF to B and C, and joining the Points B and C.

DEM.



Dem. For, the Arks, DE, EF, & FG, are equal by Conſtruction; wherefore, the Angles DAE, EAF, and FAG, are equal. But, the Angle BCA is equal to BAD, i.e. EAD; and, CBA is eq. to CAG - 13.3. conſ. BCA, CBA, BAC, being equal, reſpectively, to DAE, EAF, and FAG; the Triangle ABC is equiangular; and therefore it is equilateral - - C. 3. 9. 1.

Now, if the Ark of the Semicircle, DEFG, be divided into five equal parts, for inſcribing a Pentagon, as, before into three (in E and F) and, if A1, A2 be drawn, cuting the Circumference, at H and I; AH and HI are Sides of a Pentagon, inſcribed.

In reſpect of the Operation, I leave it to the diſcretion of the Practitioner, to take what method he moſt approves.

Granting AB to be the Side of an equilateral Triangle, and AH, HI, to be the Sides of a Pentagon inſcribed; BI is the Side of a Quindecagon, or Poligon of 15 Sides.

For, becauſe AB is the Side of an equilateral Triangle, the Ark AHB is one third part of the whole Circumference.

Alſo, AH and HI being Sides of a Pentagon, the Arks AH and HBI, are each equal to one fifth part of the Circumference, i. e. the Ark AHI is equal to two fifths, equal ſix fifteenth parts; for HI, one fifth, isequal to three fifteenths.

Now, ſince AHB is one third part of the Circumference, it contains five fifteenths; for three times five is fifteen.

Wherefore, BI is equal to one fifteenth part, and is, therefore, the Side of a Quindecagon, inſcribed in the Circle ABC.

ELEMENTS OF GEOMETRY.

BOOK V.

OF PROPORTION.

THE fifth Book of the Elements of Euclid contains the ſublime Doctrine of Proportion; which is, psrhaps, the moſt ſubtle manner of reaſoning, the moſt brief, ſolid and convincing, that the Art of Man could deviſe. It is eſſentially neceſſary in demonſtrating all the Propoſitions of the ſixth Book, which alone is ſufficient to recommend it. But, where we conſider its general and extenſive uſe, throughout the Mathematics, it is impoſſible to be diſpenſed with. The Doctrine it contains is not only uſeful in, but, it is the Criterion of almoſt every mathematical Science; infomuch, that, without the knowledge it teaches, we ſhould not be able to advance one ſtep further.

I muſt ever be of opinion that the readieſt and eaſieſt way of acquiring knowledge, is the beſt. The Doctrine of Proportion is, in ſome meaſure, born with us; it is a portion of Reaſon, implanted in us by Nature; which does not require a formal Demonſtration, ſo much as barely to be illuſtrated, by ſome familiar Examples; and which, indeed, may paſs for Axioms, for moſt of them are really ſuch.

In expreſſing and comparing Quantities, of any Species, it has been found neceſſary to make uſe of certain Symbols or Figures. Which Symbols, Marks or Characters, are entirely arbitrary, and might as well have been applied to ſignify the Qualities of Things, as Quantity. But, having been accuſtomed, from our infancy, to expreſs our Ideas of Proportion by Numbers; we no ſooner become acquainted with Numbers and the Characters applied to them, but we find it almoſt impoſſible, in our Ideas of Magnitude, or Quantity of any kind, to compare or eſtimate their Ratio, abſtracted from Numbers. In reſpect of Magnitude, as to ſolid Extenſion, or Body; we ſay, when compared, it is twice, thrice, or four times as big as another; ſo likewiſe of Extenſion, as to length, ſimply, we expreſs it by one, two, or three Yards, Chains, or Miles, &c; and ſo of any other Quantity whatever.

Hence, the mode of expreſſing Quantity, by Numbers, is the moſt natural, ſimple, and eaſy; and therefore, it is no wonder, when we find it expreſſed by other Characters, or by Lines, as it is done by Euclid and all his Commentators, in the fifth Book, that it appears, at firſt, to thoſe who have not conſidered it, quite foreign to the purpoſe; but, in reality, the difference between Lines and Numbers, in expreſſing Quantity, conſiſts, only, in being more familiarized to one than the other.

Therefore, ſince Lines, Characters, and Numbers, are only different modes of expreſſing Quantity, I muſt needs think that the beſt, which is the moſt natural and eaſy to conceive; beſides it is much more convincing; for, unleſs Lines are divided into the ſame equal Parts, whereby we may form a judgment by how much one Magnitude, or Quantity of any kind, exceeds another, we have no Idea of their Proportion; otherwiſe, than

that one does exceed or is more than the other; whereas, Numbers aſcertains the Ratio, whether it be equal or unequal.

If Characters (as A, B, C, &c.) are made uſe of, they only denote Quantity, ſimply, but in what Ratio, or Proportion, is beſt expreſſed by Numbers; algebraic Characters are not ſo expreſſive to thoſe who are not well verſed in them; they do not convey an Idea adequate to that given by Numbers, neither do they aſcertain the Ratio: and, if Lines are divided into equal Parts, to expreſs the Ratio, it is the ſame as Numbers; becauſe, we only know the Proportion by numbering the Parts.

It is eaſily known that there is Analogy, or ſimilarity of Proportion, when, in four Quantities, the firſt (i. e. any one) contains the ſecond (any other) or is contained, the ſame number of times, as a third contains the fourth, or is contained by it; whether it can be expreſſed in whole numbers, as once, twice, thrice, &c; or once, twice and a half, a third, a fourth, or any other fractional number whatever.

E. g. 6 has the ſame Proportion to 24 as 4 has to 16, or 3 to 12, each being contained four times in the other; their Ratio is, therefore, as 1 to 4. 9 has the ſame Proportion to 24 as 6 to 16; for each is contained in the other, twice and two thirds; the Ratio is as 3 to 8. So likewiſe, 27 is to 9 as 21 to 7, or 12 to 4; for, each contains the other thrice; therefore, the Ratio is as 3 to 1. 18 is to 12 as 12 is to 8, for they are, each, as 3 to 2. Thus may the Ratios of any commenſurable Quantities, whatever, be expreſſed by Numbers; and, indeed, ſuch as are incommenſurable, by approximation, are continually approaching nearer to the truth, till the deficiency is leſs than any aſſignable Quantity whatever.

Suppoſe

Suppoſe any four Terms or Characters, A, B, C, D, to repreſent four Quantities, that are analogous in their Ratio, two, and two; i. e. when A (any one) has the ſame Proportion to B (any other) as C, a third, has to D, the fourth; which is thus written, A : B :: C : D and is read thus; as A is to B, ſo is C to D. The four Points, dividing the two Pairs, denotes an equal Ratio between them.

The whole buſineſs, of this fifth Book, is to ſhew, what various ways they may be changed, and ſo ordered amongſt themſelves, that the Proportion ariſing, on both ſides, may ſtill be equal or analogous; which, will admit of great variety; ſeing, the Ratios of ſeveral Quantities, that are analogous, may be expreſſed by the ſame Numbers; as, 2 to 5 expreſſes, equally, the Ratio of 6 to 15, of 8 to 20, and of 14 to 35; each being contained in the other, reſpectively, twice and one half.

Therefore, either by Alternation, Inverſion, Compoſition, Diviſion, converting, or mixing the Terms, the Ratio of each, may, in each change, be expreſſed by the ſame Numbers; & conſequently, there is always equality of Proportion. Nor indeed, is it poſſible it ſhould be otherwiſe, if the Terms, on both ſides, be ordered & changed after the ſame manner.

Wherefore, I am and ever was of opinion, notwithſtanding what the learned Dr. Barrow, in his mathematical Lectures, and Dr. Keil, in his Preface, ſays to the contrary, and the high encomiums which Cunn, in his Preface, beſtows on Euclid's Demonſtrations of the fifth Book; that, it is involving a thing, in itſelf not very intricate, if not in darkneſs and obſcurity, at leaſt in perplexity; ſeeing that, half, or more than half, of the Propoſitions may be diſpenſed with, or paſs for Axioms; the remainder being ſufficient, towards ataining a full Demonſtration of the whole.

DEFI-

DEFINITIONS.

Def. I. QUANTITY is whatever may be meaſured or numbered, eſtimated or compared, in reſpect of more or leſs.

As Magnitude or Body; Extenſion, or length ſimply; Weight, or Gravity, Meaſure, Time, Motion, &c.

N. B. One Surface may be equal, in area, to twice, thrice or four times the area of another Surface; or a Line may have twice or thrice, &c. the length of another Line. Alſo, if one Body moves, uniformly, through twice, thrice, or four times the Space, through which another Body moves, in equal Time, it will move with twice, thrice, or four times the velocity. All which, come under the Denomination of Quantity, and may be expreſſed or repreſented by various Symbols or Characters; or by Right Lines, of twice, thrice or four times the length, one of another; or by Numbers; as 6 contains 3, 2, or $1\frac{1}{2}$, twice, thrice, or four times.

Quantities, being compared together, are of two kinds, *viz.* commenſurable, and incommenſurable.

Commenſurable Quantities are ſuch as have a common meaſure; that is, ſuch Quantities as may be divided into the ſame equal Parts, or into Parts of the ſame magnitude.

Two Quantities are commenſurable, when ſome determinate Quantity may be found, which, being taken, or multiplied, certain numbers of times, is equal to either, without deficiency or exceſs.

Incommenſurable Quantities are ſuch as no other Quantity can meaſure; i. e. there cannot be found any determinate Quantity, how ſmall ſoever, which, being multiplied, will be equal to each of the other; but that there will be a deficiency or exceſs, in one or the other.

N. B.

N. B. Any two Quantities, whoſe Proportion, to each other, can be expreſſed byNumbers, are commenſurable; two Quantities, whoſe Ratio is ſuch as cannot be expreſſed in Numbers, are ſaid to be incommenſurable to each other.

Def. II. An ALIQUOT, or equal Part, is that, when a leſs Quantity is contained any number of times, preciſely, in a greater; i. e. when a leſs Quantity, being taken or multiplied any number of times, is equal to a greater.

Thus; an Inch, or a Foot, is an Aliquot part of a Yard or Fathom; or an Ounce of a Pound, &c; 3 is an Aliquot part of 9, 12, or 15, &c.

Def. III. MULTIPLE. That Quantity is called a Multiple, in reſpect of another Quantity, when it contains, exactly, or is equal to the other, being taken any number of times; then, the leſs is ſaid to meaſure the greater.

Thus; a Foot is a Multiple of an Inch, of two, three, four or ſix Inches, but it is an Aliquot part of a Yard, &c.

A Yard is a Multiple of a Foot or of an Inch, &c. but it is an Aliquot of a Fathom or Furlong, &c.

Def. IV. RATIO, or PROPORTION,* is a mutual habitude or relation of Quantities, of the ſame kind, in reſpect to more or leſs.

Quantities are ſaid to have Ratio to one another, which, being multiplied, can exceed each other.

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* This Definition is rejected, by ſome, as ungeometrical. Profeſſor Simſon ſays, after a long quotation from Dr. Barrow's 18th Lecture, "that he fully believes, the 3rd and 8th Definitions of the 5th Book (the 4th and 6th of this) are not Euclids, but are added by ſome unſkilful Editor." I muſt freely own, that I am at a loſs to conceive why it is they cavil at them; which, to me, ſeem both proper and neceſſary.

Dr. Barrow ſays, that, "Euclid had, perhaps, no other deſign in giving this Definition (which he calls metaphyſical) than to give a general, though a groſs and confuſed, notion or idea of Ratio

to

In order to which, it is neceſſary that they be of the ſame ſpecies or kind, e. g. A Line (having no breath) cannot, by multiplying, be compared with, or be equal to the ſmalleſt ſurface. Alſo a Surface, (having no thickneſs) cannot, by multiplying, be equal to, or compared with a Solid. So neither can Weight be compared with Time, or Motion, &c. Wherefore, ſuch Quantities as are not of the ſame kind, are heterogeneous, and can have no Ratio to one another.

N. B. The Ratio of two Quantities is not expreſſed by the deficiency or exceſs of the Antecedent to the Conſequent; although they may be compared by them, it is not their Ratio.

RATIO, is the Value of the Antecedent Term in reſpect of the Conſequent. (See the next Definition).

Thus, If A is to B as 3 to 5; the Ratio is $\frac{3}{5}$ or 3 fifths of the Conſequent. But, if B be taken for the Antecedent, the Ratio is greater, viz. as 5 to 3; i. e. $\frac{5}{3}$ or 5 thirds, which is one, and two thirds of the Conſequent; whereas, in the former Ratio, the Antecedent, is two fifths deficient of its Conſequent.

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to beginners" grant it; was it not neceſſary, then, to give ſuch a general Idea? which, if it be metaphyſical, cannot, I preſume, with propriety, be called groſs, though it may be confuſed; as, in my Opinion, is every thing relative to Metaphyſics, for, we can only reaſon, on ſuch abſtracted ſubjects, from Analogy, of imaginary Beings with coporeal; which do not, properly, admit of Analogy.

He ſays, further, that nothing in Mathematics depends on it, (the 6th) and that, both might well be ſpared, without any loſs to Geometry." I am aſtoniſhed at ſuch an Aſſertion; does nothing depend on Proportion, or Ratio, and Analogy of Ratios? can we (as he ſays we ſhould principally) attend to thoſe which follow, before we know what is meant by Ratio? can we have any notion of any one, from Alternate Ratio to equality of Ratios, before we know what is meant by Ratio, ſimply? which, in itſelf is undoubtedly metaphyſical; yet, we cannot have any notion of Ratio in abſtract; as the Doctor takes great pains (in his 20th Lecture) to convince us, that Ratios are not Quantities; who ever ſuppoſed it? at the ſame time, I find it impoſſible to form an idea of Ratio, abſtracted from Quantity, which is the Eſſence of Ratios.

A Ratio

Def. V. ANTECEDENT and CONSEQUENT are the Terms of a Ratio.

In every Ratio there muſt neceſſarily be two Terms, denoting two Quantities. The Antecedent is that which is firſt named, and being referred to, or compared with another Quantity, the latter is called the Conſequent.

Thus, A : B, or B : A ſignifying, that A has ſome Ratio to B, or B to A, (which it neceſſarily muſt if they be ſuppoſed to repreſent two Quantities of the ſame kind) A or B is the Antecedent, B or A is the Conſequent, according, as they are taken, firſt or laſt.

A Ratio has no exiſtence, either phyſically or metaphyſically, without Quantity; the Term, Ratio, implies Quantity; where is the Ratio to exiſt, if not in Quantity? can we conceive a Ratio between non-exiſtences? the ſuppoſition is abſurd, and cannot be. Therefore, I look on the Definitions of Ratio, and Analogy of Ratios, to be abſolutely neceſſary in Geometry, that they were given by Euclid, and that his Elements of Proportion would be imperfect without them.

Some Geometers make a diſtinction between Ratio and Proportion; for, Dr. Barrow, in his 4th, and Mr. Stone, in his 8th Definition, ſay, "Proportion is a ſimilitude of Ratios," which, if Ratio and Proportion be the ſame, (as the Doctor affirms, in a Note, annexed) is an abſurd Definition; viz. Ratio is a ſimilitude of Ratios. Dr. Keil, in his 8th Definition, calls Proportion Analogy, and ſays, "Analogy is a ſimilitude of Proportions;" and Profeſſor Simſon ſays, "Analogy, or Proportion, is the ſimilitude of Ratios."

Now, in my opinion, the diſtinction between them is ſo very nice, that it requires very acute diſcernment to diſcover the difference. What is by them called Proportion is, properly Analogy, i. e. when the Ratios are equal or ſimilar; for, Dr. Keil, in his 4th Definition, ſays, "Magnitudes are ſaid to have Proportion to one another." Mr. Stone ſays, "Magnitudes are ſaid to have a Ratio to one another." Dr. Barrow, in his fifth, changes Magnitudes for Numbers, but makes uſe of the Term Ratio. So likewiſe, in the ſixth Definition; Keil ſays, "Magnitudes that have the ſame Proportion;" Simſon and Stone ſay, in the 6th "Magnitudes which have the ſame Ratio, are called Proportionals;" conſequently, they all mean the ſame thing, by Ratio and Proportion.

Def. VI. ANALOGY,* of RATIOS, is a ſimilitude or equality of Ratios. i. e. When the Antecedent of one Ratio contains its Conſequent, equally, as the Antecedent of another Ratio contains its Conſequent.

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* It was not the Deſign, of thoſe Geometers who made uſe of this Term, to define Analogy, in abſtract, but only, to explain what is meant by analogy of Ratios (between two and two Quantities) which is a very proper and expreſſive Term, for equality of Ratios; it was neceſſary, that ſome one or other ſhould be made uſe of. Analogy, applied to Ratios, is, at leaſt to me, very expreſſive of the thing meant by it; much more ſo, in my Opinion, than Proportion; which is only another Term for Ratio, or Ratio for Proportion; which, Terms, are ſynonimous, and equally expreſſive. To what purpoſe are all the various changes, by alternation, inverſion, &c. if we take away Analogy? all the uſe that is or can be made of them is, only, to ſhew, that the reſult of theſe changes is Analogy; the chief, if not the only way of reaſoning from Proportion, in general; conſequently it is not an unneceſſary Term.

If we muſt needs be caviling, there are other Definitions which are much more exceptionable; as the fourth of Euclid, viz. "Magnitudes are ſaid to have a Ratio to one another" &c; in reality, it is not a Definition of any thing, but is only meant to acquaint us, what Quantities can have Ratio to each other; which common ſenſe has amply provided for: who would ever think of comparing Time with Gravity, Meaſure, &c; or with any corporeal ſubſtance, Superficies or Solids?

The fifth is not, properly, a Definition, but intended, only, as a ſign of Analogy (ſee the Remark on it, at the end of this 5th Book). The 9th is leſs ſo than either of the former, viz. Proportion (or Analogous Ratio) conſiſts of three Terms, at leaſt. What does this define? nothing; can any Perſon be at a loſs to know, that there muſt needs be three Terms, who has had Analogy of Ratios defined, where he always finds four? but, one of the four, is ſometimes taken twice. I therefore, look on it as ſuperfluous and unneceſſary, ſeing it is a neceſſary conſequence in Analogy; yet, it is not improper, by way of Nota bene or Remark; but there could be no impediment, to the attaining of the reſt, if it was never mentioned. Therefore, it may, very ſafely, be bloted out of the Vocabulary of Definitions; for, to uſe Dr. Barrow's own words, Proportion would ſuſtain no loſs by it; and I do verily believe, that this was never given by Euclid, as a Definition.

Or, when the Antecedents, of both Ratios, are contained equally in their reſpective Conſequents.

Thus; 9 has the ſame Ratio to 3, as 6 to 2; for each is as 3 to 1, i. e. each Antecedent contains its Conſequent thrice; therefore, there is the ſame Ratio or Proportion between them, and they are ſaid to be Analogous.

Alſo, 6 is to 9 as 2 to 3, for each Antecedent is contained once and a half in its Conſequent.

When the ſecond Term has not the ſame Proportion to the third, as the firſt has to the ſecond, or the third to the fourth, they are ſaid to be directly or diſcretely proportional.

Thus, 4:6::8:12, directly or diſcretely; but, 4:6::9:13½ is both directly and continual. (See Def. 7.)

When one Antecedent contains its Conſequent, or is contained in it, more than the other Antecedent contains or is contained in its Conſequent, though by ever ſo ſmall a part, the Ratio is unequal; and they are called, ſimply, UNEQUAL RATIOS.

Analogy of Ratios is thus expreſſed; A:B::C:D.

The firſt and third Terms, A and C, are the Antecedents, and may be taken for any Quantities whatever; B and D, the third and fourth, are the Conſequents.

Wherefore, if A be ſuppoſed, in Numbers, to repreſent 9, and C, 3; and, if B be equal 6, it will be thus, 9:6::3:2; conſequently, D will repreſent 2. But, if B alſo repreſents 3, then will D expreſs or ſignify but 1, viz. 9:3::3:1.

Again; if B repreſent a greater Quantity than A, D will alſo be greater than C; thus, 9:12::3:4; or 9:15::3:5.

The three firſt Terms being fixed or known Quantities, and in a certain order, the fourth is a neceſſary conſequence.

N. B. In Analogy of Ratios, there muſt neceſſarily be three Terms, ſeing that there is ſimilitude or equality of Ratios, (and each Ratio conſiſts of two Terms, the Antecedent and Conſequent;

ſequent; Def. V.) But, ſince the Conſequent of the firſt Ratio may, alſo, be the Antecedent of the ſecond (as A:B::B:C) three Terms are ſufficient to conſtitute Analogy; in which caſe, it is neceſſary that they are all of the ſame ſpecies or kind.

But, when there are four Terms (A:B::C:D) the firſt Ratio (A:B) may be of a different ſpecies from the other (C:D) for, there is the ſame Ratio between a Foot and an Inch, two, three, or four Inches, &c. as between a Pound, and one, two, three, or four Ounces, (Troy Weight) conſequently, there may be Analogy of Ratios, between heterogeneous Quantities.

Def. VII. CONTINUAL RATIO is when, in three or more Quantities, the firſt is to the ſecond, as the ſecond to the third, as the third to the fourth, &c.; or, more properly, the third is to the fourth, as the ſecond to the third, and that, as the firſt is to the ſecond.

Thus. In four or more Quantities, A, B, C, D, if C has the ſame Proportion to D, as B has to C; and, if B has alſo the ſame Proportion to C, as A has to B, they are then in continual Proportion, or geometrical Progreſſion; and are thus writ, or ſymbolized; A:B:C:D ÷.

N. B. Each Term being twice taken, except the firſt and laſt, are, therefore, both Antecedent and Conſequent, i. e. the ſecond Term is a Conſequent in reſpect of the firſt, and the third of the ſecond; but they are, alſo, Antecedents in reſpect of the third and fourth Terms.

In continual Ratio, all the Terms are, neceſſarily, of the ſame ſpecies or kind. i.e. The Quantities muſt be homogeneal.

If A and C, the two firſt Terms, or D and C, the two laſt, are any fixed or determinate Quantities, all the reſt are determinable; which, in Numbers, is obvious.

Let A be 4, and B 6; it will then be thus, 4:6:9:13½ ÷ &c. each conſequent Term containing its Antecedent once and a half. But if A be = 3, and B 6, it will be 3:6:12:24 ÷ the Ratio being as 2 to 1, or double.

Or, if A repreſents 2, and B 6, then 2:6:18:54 ÷ &c. the Ratio being triple, or as to 3 to 1. So that, whatever Ratio there is between the firſt and ſecond, that Ratio is continued.

Therefore,

Therefore, when the Ratio is increaſing (or rather when the Quantity is increaſing, for the Ratio is the ſame, throughout) the Quantity of C, D, &c. is determined either by Addition or Multiplication; as in a decreaſing Ratio, (i. e. when the Antecedent is the greater Quantity) by Diviſion or Subtraction.

Def. VIII. PROPORTIONALS are ſuch Quantities as are in the ſame Ratio; diſcretely or continual.

Proportionals conſiſt, at leaſt, of three Terms, or Quantities; which are uſually ſymbolized by Right Lines; or by Characters, as A:B::C:D, the Ratio being determined by Numbers.

Def. IX. MEAN PROPORTIONAL, and THIRD PROPORTIONAL.

When there are three Terms, only, or, if three Terms are Proportionals, as A, B, C, they are in continual Proportion; thus, if A is to B, as B to C, in equal Ratio; then, the middle Term, B, is a Mean, between the other two; and is, therefore, called a MEAN PROPORTIONAL.

The firſt and laſt Terms are the Extremes of the Proportionals; either of which, A or C, when there are but three, is called, a THIRD PROPORTIONAL, in reſpect of the other two.

Hence it is evident, that, in continual Proportion, of four or more Quantities, each Term, except the firſt and the laſt, is a Mean, between its Antecedent and Conſequent.

N. B. If two Terms or Quantities are given, and a third be required, in the ſame Ratio to either of the given Terms as that has to the other, the Quantity found is a third Proportional.

Def. X. EXTREME and MEAN PROPORTION is, when any Quantity is ſo divided, into two Parts, that, the leſſer, the greater, and the whole, are in continual Ratio.

A Right-

A Right Line is ſaid to be divided in extreme and mean Proportion; when, being cut into two unequal Parts, the Ratio of the whole Line, to the greater Segment, is the ſame as of the greater Segment to the leſs.

If the Line AB be cut in C.

Then, if the whole Line, AB, is to AC as AC is to CB; AB is divided in extreme and mean Proportion.

Def. XI. A Fourth Proportional.

If four Terms, A, B, C, and D, are Proportionals; whether they are analogous, only, or in continual Proportion, either of the Extremes, A or D, is a fourth Proportional, in reſpect of the other three.

N. B. If three Terms or Quantities are given, and a fourth is required, in the ſame Ratio, to any one of the given Quantities, as the remaining two have, the one to the other; the Quantity, ſo found, is a fourth Proportional.

Def. XII. Duplicate and Triplicate Ratio.

If four, or more Terms, A, B, C, D, are in continual Proportion; the firſt, A, is ſaid to have, to the third, C, a duplicate Ratio of A to B, the firſt to the ſecond.

And the Proportion of A to D, the firſt to the fourth, is triplicate of A to B; and ſo on.

Def. XIII. Alternate Ratio is the mutation of the two middle Terms, of four proportional Quantities;* i. e. when Antecedent is compared with Antecedent, and Conſequent with Conſequent; and they are called homologus, corational, or correſponding Quantities.

Thus, A:B::C:D, directly. — 27:18::6:4.
Then, A:C::B:D, alternately — So, 27:6::18:4.

* Alternate Ratio cannot exiſt or take place, unleſs both Ratios are of the ſame ſpecies; ſeeing, there is no Ratio or compariſon between

N. B. When the Product, ariſing from the multiplication of the two middle Terms, is equal to that of the two Extremes, it is a certain proof of Analogy, conſequently, it will be the ſame, whether they are taken directly or alternately.

Def. XIV. INVERSE RATIO is when the Conſequent is taken for the Antecedent, and compared with the Antecedent as a Conſequent.

In this Caſe, the middle Terms become the extremes.

If A:B::C:D directly. — 27:18::6:4.

Then, B:A::D:C inverſely. — So, 18:27::4:6.

Def. XV. COMPOUNDED RATIO is when the Antecedent and Conſequent, being taken or added together, as one Quantity, are compared either with the Antecedent or with the Conſequent.

If A:B::C:D; then, A+B:A, or B,::C+D:C, or D.

6:9::2:3; then, 15:6, or 9,:: 5:2, or 3.

Def. XVI. DIVIDED RATIO is when the difference, between the Antecedent and the Conſequent, is compared either with the Antecedent or the Conſequent.

If A:B::C:D. Then, whether A be greater than B, or B than A; the difference, being taken, is compared either with A or B; as alſo, the difference between C and D, is compared with C or D.

Thus, A−B, or B−A,:A, or B,::C−D, or D−C:, C or D,

15:6::5:2; then, the difference, 9,:6, or 15,::3:2, or 5.

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between Quantities that are heterogeneous. (ſee Def. IV.). For, if four Quantities are in analogous Ratio, (A:B::C:D,) it is not neceſſary that the firſt pair (A, B,) ſhould be of the ſame kind as the laſt (C and D.) But, in Alternate Ratio, the Antecedent of one Ratio is compared with the Antecedent of the other, and Conſequent with Conſequent; wherefore, it is abſolutely neceſſary that both Ratios are of the ſame kind.

N. B. In the ſeven following Definitions, it is the ſame whether they are of the ſame kind or not; i. e. each pair may be of a different kind and heterogeneous to the other.

Def. XVII.

Def. XVII. CONVERSE RATIO is when the Antecedent is compared with the difference between the Antecedent and the Conſequent.

If A:B::C:D. Then, whether A exceeds B, or B exceeds A; it will be thus; A:A—B, or B—A,::C:C—D, or D—C. 12:27::4:9; and, 12 or 27,:15 (the difference)::4, or 9:5.

Def. XVIII. MIXED RATIO is when the Antecedent added to the Conſequent, as one Quantity, is compared with the exceſs, by which the one exceeds the other;

And, vice verſa, when the difference, between them, is compared with the ſum of both.

If A:B::C:D; then, A+B:A—B, or B—A,::C+D :C—D, or D—C. 7.3::21:9; then, 10:4::30:12, the Antecedent being the greater Quantity.

Def. XIX. ORDINATE RATIO of EQUALITY.

If there be three or more Quantities in one Rank or Order, in any Ratio whatever, and as many in another Rank, in the ſame Ratio, comparing two and two; i. e. as the firſt is to the ſecond, in one Rank, ſo is the firſt to the ſecond, in the other; and, as the ſecond is to the third, ſo is the ſecond to the third, &c.; it will then be as the firſt is to the laſt, in one Rank, ſo is the firſt to the laſt in the other; or (comparing, the Extremes) as the firſt is to the firſt, ſo is the laſt to the laſt.

In the two Ranks A, B, C, D, and E, F, G, H;
If, A:B::E:F, and B:C::F:G; alſo, as C:D::G:H.
Then, as A:C, or D,::E:G or H; alſo, as A:E::C:G, or D:H.

Let the firſt Rank be as 3:7:5:8; and let E be 9
Then, F will be 21, G will be 15, and H 24.
For, as 3:7::9:21; and as 7:5::21:15; alſo, as 5:8::15:24.
Then 3:5, or 8, as 9:15, or 24; and as 3:9::5:15, or 8:24.

Ratio

Ratio of equality * is of two kinds, ordinate and inordinate.

Ordinate Proportion is what is already defined; or, when there are, at firſt, but four Quantities, in analogous Ratio, either diſcretely or continual; and other Quantities are aded, after the ſame manner, and in equal Ratio.

For example; let the Ratio of C:D be as A:B.

Now, if other Quantities be added, on both ſides, in ſuch wiſe, that the Ratio is the ſame, ordinately, there is ſtill equality.

Firſt, let the Quantities added be Antecedents;
If A:B::C:D; then, let E be to A as F to C;
or, let them both be Conſequents, viz. as B:G::D:H.

It is manifeſt, that, in either Caſe, the Ratio is ſtill the ſame; being taken progreſſively.

For, as E:B, or A:G,::F:D, or C to H;
alſo, as A:B, or C:D,::E:F, or G to H; in either.

Def. XX. INORDINATE RATIO of EQUALITY is, when in equal Ratios of two Quantities each; as the Conſequent of one Order is to ſome other Quantity, ſo is another Quantity to the Antecedent of the other Order.

Then, as the Antecedent, of the firſt, is to that other Quantity, ſo is the other Quantity to the Conſequent of the other Order.

If A:B::C:D; and, if B:E::F:C; then, A:E::F:D.
3:4::21:28; and, 4:14::6:21; then, 3:14::6:28.
Again, if E:G::H:F; then, A:G::H:D.
For, 14:7::12:6; then, 3:7::12:28.

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* By Ratio of equality is not meant equality of Ratios ſimply; but, in two Ranks of Quantities, whoſe Ratios are ordinate or inordinate, the Ratios are equal, at equal Diſtances from the Extremes.

Def. XXI. COMPOSITION of RATIOS.*

In any Number of Quantities, and in any Order, whatever; the Ratio of the first to the last is equal to that which is compounded of all the intermediate Ratios.

If A, B, C, and D, represent as many Quantities; whatever is the Ratio of A to B, of B to C, and of C to D; the Ratio of A to D, is equal to the Ratio compounded of them all.

Thus, if A is to B as 3 to 5, B to C as 5 to 8, and C to D as 8 to 6.

Then, A : D :: 3 : 6; compounded of $\frac{A}{B}\ \frac{3}{5}$ of $\frac{B}{C}\ \frac{5}{8}$ & of $\frac{C}{D}\ \frac{8}{6}$

i.e. Multiply all the Antecedents into one another; $3 \times 5 \times 8 = 120$; and divide the Sum, by that of all the Consequents; $5 \times 8 \times 6 = 240$, then is A to D, as 120 to 240, or as 3 to 6; i.e. as 1 to 2, or $\frac{5}{10}$.

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* This last Definition is generally given in the sixth Book, for which I cannot conceive a reason. The Doctrine of Proportion is a distinct subject, and therefore, all which relates to it should be together, in one Book; which, according to Euclid, is the fifth: to which (in the use and application of it, in the sixth and other Books,) we may refer on all Occasions.

Dr. Barrow, indeed, gives nearly the same thing, in both. In Def. 20. B. 5. he says, "any number of Magnitudes being put; the proportion, of the first to the last, is compounded out of the proportion of the first to the second, the second to the third, and the third to the fourth, &c. till the proportion arise."

The proportion, which is between the first and the last, will, and must necessarily, arise, by the composition of all the intermediate Ratios, which is evident from the manner of compounding them; nor is it possible to be otherwise; seeing that, each of the intermediate Quantities is both Antecedent and Consequent. Therefore, the Ratio emerging at last, seeing, it depends entirely on the first and last, must ever be equal to the Ratio of the first to the last; let the intermediate Quantities be ever so many, and in what Proportion soever.

According

According to Euclid, there are ſix various ways of reaſoning, from Proportion, including ordinate and inordinate Ratio in that of Equality; which, in effect, is all one. Inordinate Ratio differs from ordinate, in nothing but the diſpoſal of the laſt Quantity; for, whether it be conſidered and compared as an Antecedent or as a Conſequent, by placing it firſt or laſt; the Ratio, in either caſe, being the ſame, it will ſtill be as the firſt to the laſt, ſo is the firſt to the laſt; but not, one Quantity, (added) to the other, as in the former caſe, where the Quantities, added, are both Antecedents, or both Conſequents.

But there is another way of reaſoning by Proportion, diſtinct from the reſt, which is not in Euclid; viz. mixed Ratio, (Def. XVIII.) that is, comparing the ſum of the Antecedent and Conſequent with their difference. In reſpect of Converſe Ratio, it differs in nothing from inverſe, which includes all Converſe Ratios whatever; ſince, all Ratios, that are analogous, are inverſely ſo, or converſely, which is the ſame thing. Nor do I ſee any Reaſon, why the 15th Definition, of Compounded Ratio, has not a converſe, as well as the 16th of Divided Ratio; ſeeing that, one holds good as well as the other; and is as neceſſary to be known, it being as frequently made uſe of as the other.

That the manner, and the order, of the ſeveral changes may be ſeen at one view, I have given them over again; with the converſe of each, in the following abſtract; keeping the ſame Ratio throughout the whole. In which five Caſes, viz. alternating, inverting, compounding, dividing and mixing, with their converſe, together with equality, ordinate and inordinate, is contained all the variety in which the Terms, of analogous Ratios, may be changed and ſtill be analogous; and which, anſwers almoſt every purpoſe, of reaſoning from Proportion.

If A:B::C:D, directly, as 3 to 2; thus, 21:14:: 9:6.
Then, A:C::B:D, alternately. - - - 21: 9::14:6.
Alſo, B:A::D:C, inverſely. - - - - 14:21:: 6:9.
And, A+B:A, or B,::C+D:C, or D. } by Compoſition.
35 : 21, or 14, :: 15 : 9, or 6. }
Alſo; A, or B, :A+B ::C, or D, :C+D. } converſely.
21, or 14, : 35 :: 9, or 6, : 15 }
And, A—B : A, or B, ::C—D :C, or D. } by Diviſion.
7 :21, or 14,:: 3 : 9, or 6, }
Alſo; A, or B, : A—B::C, or D:C—D } converſely.
21, or 14: 7 :: 9, or 6 : 3 }
Again, A+B:A—B::C+D:C—D, mixing; 35: 7::15: 3.
or, A—B:A+B::C—D:C+D, converſely, 7:35:: 3:15.

Ordinate Ratio of equality.

If A : B::C:D; and, if B:E ::D: F; then, A: E::C: F.
21:14::9: 6; and, 14:42:: 6:18; then, 21:42:: 9:18.

Inordinate Ratio of equality.

If A : B::C:D; and, if B: E::F:C; then, A: E::F:D.
21:14:: 9: 6; and, 14:42:: 3: 9; then, 21:42:: 3: 6.

Thus, having ſhewn all the variety of changes, and, that the Ratios emerging from each, in numbers, are equal or analogous; there remains, one might reaſonably ſuppoſe, nothing more to be done. For, after this knowledge once obtained, of what uſe can it be to perplex the Student, with a tedious Demonſtration of what, in Numbers, is clear and manifeſt; and which, I am perſuaded, needs no further Demonſtration. Nor do I ſee, excepting Alternate Ratio, (which is, alſo, ſufficiently evident, in Numbers) that any of the reſt require Demonſtration; or are made more manifeſt by it. For, the Ratios of all commenſurable Quantities may be expreſſed by Numbers; and it is ſufficiently manifeſt, that, by Analagy, it muſt alſo hold true in reſpect of Quantities, which are incommenſurable.

It

It is evident, that, ſince A:B::C:D, whether the Ratio of A to B be commenſurable or incommenſurable, the Ratio of C to D being the ſame, as A to B, they will ſtill be analogous in every change.

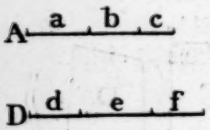
E. g. Suppoſe the Ratio of A to B and of C to D be as the Side of a Square to the Diagonal; or, as the Diameter of a Circle is to its Circumference, &c.

Now, it is manifeſt, that the Side of one Square is to its Diagonal, as the Side of any other Square is to its Diagonal; conſequently, their Ratios are equal, notwithſtanding they cannot be expreſſed in Numbers; for 3 : 5 :: 9:15, i. e. 3 : 5 :: 3:5; conſequently, 3:3::5:5, &c. Therefore, ſince the Side of one Square, is to its Diagonal, as the Side of any other Square to its Diagonal; conſequently, as Side is to Side, ſo is Diagonal to Diagonal; alſo, as the Side added to the Diagonal, is to either Side or Diagonal; ſo is any other Side added to its Diagonal, to either; likewiſe as one Diagonal is to the Side, ſo is the other Diagonal to the Side; or, as the Diagonal, leſs the Side, to either, ſo is the Diagonal, leſs the Side, to either; and thus it muſt ever be through all the variety of Changes that can be, if they are ordered and changed after the ſame manner, on both Sides.

In what I have advanced, on the Subject of Proportion, is contained the eſſence of the whole fifth Book of Euclid; and if it be clearly underſtood, the Doctrine of Proportion is already acquired; and conſequently, all that can be ſaid more of it, by way of Demonſtration, is, in a great meaſure, ſuperfluous. However, not to leave the whole Doctrine of Proportion without a Foundation, leaſt it ſhould be cenſured by the ſcrupulous, I have demonſtrated or more fully illuſtrated, ſome of the moſt eſſential, on which the whole depends; ſome of the others I have made Axioms, for being ſelf evident they require no Demonſtration.

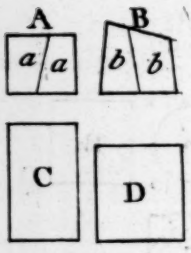
AXIOMS,

AXIOMS, or ſelf-evident PROPOSITIONS.



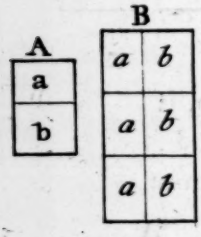
Ax. I. One Quantity is to any other Quantity, as all the Parts of one, is to all the Parts of the other.

Let A and D be two Quantities, and, let A be divided, any how, into Parts, a, b, c; alſo let D be divided any how, in d, e, and f.
Then, A : B :: a + b + c : d + e + f.



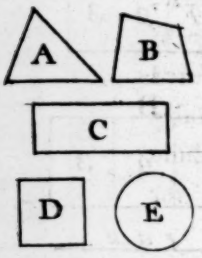
II. Equimultiples,* or equal Parts, of equal Quantities are equal.

Let A and B be equal Quantities; then, if C and D be equal twice or thrice, &c. of A and B, reſpectively; 2A=2B, or 3A=3B, i. e. C=D. Alſo, if A and B be divided into two or more equal Parts, *a a*, *b b*, &c; *a* = *b* or $\frac{1}{2}$A=$\frac{1}{2}$B, &c.



III. Any Multiple of the whole is equal to the ſame Multiple of all its Parts, taken together.

Let B be taken any Multiple of A; then, whatever Multiple B is of A, it is manifeſt, that it contains the ſame Multiple of a and b, the Parts of A and B; a, *a*, *a*, and b, *b*, *b*.



IV. The ſeventh Propoſition of Euclid.

Equal Quantities have the ſame Ratio to any third Quantity, or to equal Quantities.

For, if A and B be equal Quantities, they have the ſame Ratio to a third Quantity, C, or to equal Quantities, D and E; ſeeing that, either may be taken for the other.

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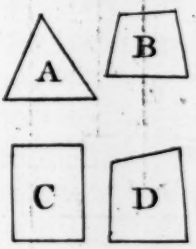
* By Equimultiples is meant, that the Quantities are taken, or multiplied, an equal number of times.

V. The

V. The ninth Propoſition of Euclid.

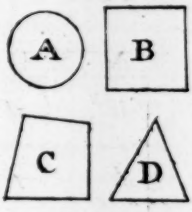


Quantities that have an equal Ratio to the ſame Quantity, or to equal Quantities, are equal.

If A and B have the ſame Ratio to C, or D; A and B are equal. Alſo, if A has the ſame Ratio to C, as B to D; if C=D, A=B.

VI. The tenth Propoſition of Euclid.

That Quantity which has the greater Ratio to a third Quantity, or to equal Quantities, is the greater Quantity. And, of two Quantities, that, to which a third Quantity has a greater Ratio, is the leſſer Quantity.

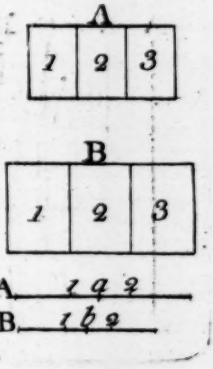


If C has a greater Ratio to B, than A has to B, C is greater than A.

Alſo, if A has a greater Ratio to D than to C, D is leſs than C.

VII. The fifteenth Propoſition of Euclid.

Quantities have an equal Ratio to their Equimultiples ; or to their equal Parts.



Let A and B be two Quantities, in any Ratio whatever; and let them be taken an equal number of times. Or, let them be divided into the ſame number of equal Parts; The wholes of A and B are equimultiples of thoſe Parts,

A : 2A or 3A, :: B : 2B or 3B. i.e. 1 : 2 or 3 :: 1 : 2 or 3.
Or, as $\frac{1}{2}$A, or $\frac{1}{3}$A, : A : : $\frac{1}{2}$B, or $\frac{1}{3}$B, : B.
Alſo, A : A*a* ($\frac{1}{2}$A) or A1, : : B : B*b* ($\frac{1}{2}$B) or B1.

VIII. Quantities are in the ſame Ratio, to each other, as their Equimultiples ; or, as their equal Parts.

Let

Let A and B be any Quantities, divided into equal Parts, as before, at *a* and *b*, or 1, 2, 3; alſo, let C and D be any equimultiples of A and B.

Then, A : B :: 2A : 2B, or as 3A : 3B; i. e. as C : D.

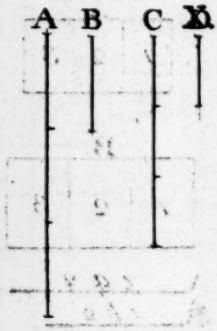
For C and D are any equimultiples of A and B; wherefore, as often as A contains B, or is contained in B; ſo often 2 A contains 2 B, or is contained in 2 B; i. e. as C contains D, or is contained in D; conſequently, A : B :: C : D.

Alſo, as A : B :: A *a* ($\frac{1}{2}$A) : B *b* ($\frac{1}{2}$ B)
or, as A 1 to B 1. ($\frac{1}{3}$ A to $\frac{1}{3}$ B.)

IX. If two Quantities be divided into Parts of equal magnitude (conſequently equal amongſt themſelves) then, one Quantity is in the ſame Ratio to the other, as the number of Parts, in one, to the number of Parts, in the other.

Let A and D be two Quantities, divided into equal Parts, *a*, *b*, and *c*; *d*, *e*, *f*, and *g*;

Then, A : D :: 3 : 4; or, as 6 to 8, &c.
i. e. as $a + b + c$ is to $d + e + f + g$.

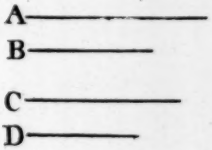


X. If Quantities are proportional; the firſt to the ſecond, as a third is to a fourth, &c; then, if the firſt be any multiple or equal part of the ſecond, the third is an equimultiple or equal part of the fourth, &c.

For Quantities are to each other, as the number of Parts in one, to the number of Parts in the other. - - - - by the 9th.

Therefore, if A : B :: C : D; A contains, or is contained in B, as often as C contains, or is contained in D. - - - - - Def. 6.
otherwiſe, the Ratios are not equal, or analogous.

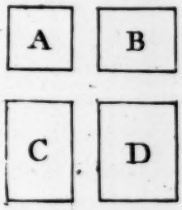
AXIOM XI. In four proportional Quantities, i. e. when any one is to another, as a third is to the fourth, (Def. 6.) whether the firſt be equal to, greater, or leſs than the ſecond, the third is alſo equal to, greater, or leſs than the fourth.



If A:B::C:D; then, if A be equal to B, C=D; and, if A be greater, or leſs than B; C is, alſo, greater, or leſs than D.

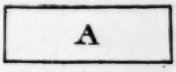
For, A has the ſame Ratio to B, as C to D;

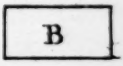
And, a greater Quantity cannot have the ſame Ratio to a leſs, as a leſs to a greater. - Def. 6.



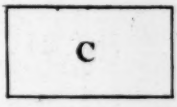
XII. The fourteenth Propoſition of Euclid.

If four Quantities are proportional; then, if the Antecedent of one Ratio be greater than the Antecedent of the other, the Conſequent of the firſt is alſo greater than the Conſequent of the other; if the firſt are equal, the other are alſo equal; and, if leſs, leſs.

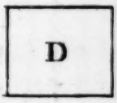




For, A:B::C:D; and, if A be equal to, greater, or leſs, than B, C is alſo equal to, greater, or leſs than D. - - - - - - 11



Wherefore, if A=C, B=D (Ax. 5.) and conſequently, if A be greater or leſs than B, C is, neceſſarily, greater or leſs than D.



XIII. The eleventh Propoſition, of Euclid.

Ratios that are equal to the ſame Ratio, or to equal Ratios, are equal between themſelves.

This follows from the third Axiom of Book 1ſt, by ſubſtituting Ratios for Quantities.

XIV. The thirteenth Propoſition of Euclid.

If four Quantities are proportional, the firſt to the ſecond, as the third to the fourth; and if the Ratio of the third to the fourth, be greater or leſs than a fifth Quantity has to a ſixth; the Ratio of the firſt to the ſecond, is alſo greater or leſs than the fifth to the ſixth.

If A:B::C:D; and, if the Ratio of C to D be either greater or leſs than E to F; the Ratio of A to B is alſo greater or leſs, than E to F.

For, the Ratio of A to B is equal to the Ratio of C to D; therefore, their Ratios are the ſame, in reſpect to any other.

Note. By miſtake, the 7th and 8th Axioms are miſplaced. But, as either is the other, alternately, it is of no conſequence; ſave only, that the firſt part of the 8th is, properly, Euclid's fifteenth Propoſition.

POSTULATES.

1. Grant, that equal Ratios may be taken, one for the other.

2. Grant, that any Quantity may be divided into any number of Parts, equal to one another; or as any other Quantity is divided.

3. Grant, that a Quantity may be taken or aſſumed, in any Ratio to any given Quantity.

The aſſumption of this laſt Axiom is not allowed by ſome Geometers; notwithſtanding, to me, it appears full as poſſible, as that equimultiples, or equal parts of Quantities may be taken; by one or other of which Suppoſitions, they demonſtrate the whole fifth Book.

THE

THEOREM I. 1 Euclid.

If Quantities, which are equimultiples or equal parts of other Quantities, be added into one Sum, or Quantity; the ſame multiple or part, which one Quantity is of the other, reſpectively, the whole is of the whole.

Let A, B, and C be equimultiples, or equal parts, of D, E, and F, reſpectively; i. e. whatever multiple, or part, A is of D, let B be the ſame multiple, or part, of E, and C of F.

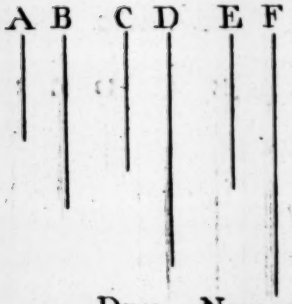
Then, A, B, and C, added together, into one Sum, is the ſame multiple or part of D, E, and F, added together, as A is of D, B of E, or C of F.

Dem. For, let A be equal to 2D, or 3D; then, B = 2E, or 3E, &c. conſ. A+B+C = D+E+F, taken twice, or thrice, &c. But, Quantities are to each other, as the number of Parts, in one, is to the number of equal Parts, in the other; Ax. 9. wh. as A:D, B:E, and C:F, ::A+B+C:D+E+F. Conſequently, as often as A contains D, or is contained in D, &c. ſo often the whole of A+B+C contains, or is contained in, D+E+F. Therefore, &c. Q. E. D.

For, if either be multiples of the other; the latter is, conſequently, equal parts of the former.

THEOREM II. 12 Euclid.

If any number of Quantities, are Proportionals, (i. e. being taken two and two, their Ratios are analogous) then, as any one of the Antecedents is to its Conſequent, ſo is the ſum of all the Antecedents to the ſum of all the Conſequents.



Let A, B, C, D, E, and F be proportional Quantities; of which, let A be to B, as C is to D, and as E to F.

I ſay, as A is to B, or C to D; ſo is A, C, and E, added together, to B, D, and F, together.

DEM. Now, A:B::C:D, and, as E:F, therefore, their Ratios are equal, - - - - - Def. 6.
Wherefore, if A=B, C=D, and E=F - - Ax. 11.
conſequently, A+C+E=B+D+F. - - Ax. 6. 1.
therefore, as A:B, or as C:D, ::A+C+E:B+D+F.

Again, if A be any multiple, or equal part, of B; then, C is an equimultiple, or equal part, of D, & E of F - Ax. 10.
wherefore, A+C+E is the ſame multiple, or equal part, of B+D+F, as A is of B, as C of D, and as E of F - Th. 1.

But, Quantities have the ſame Ratio, to each other, as their Equimultiples, or as their equal Parts. - - Ax. 8.
Th. as A:B, as C:D, or as E:F, ::A+C+E:B+D+F.

Now, becauſe there are ſuch Quantities, which cannot be equimultiples or equal Parts, one of the other, this Demonſtration is not poſitive, in reſpect of ſuch Quantities; and, notwithſtanding they may be divided into the ſame number of equal Parts, yet it will be found no eaſy matter, to give perfect Demonſtration, ſeeing that, equal parts are in the ſame Ratio to each other as the wholes. Ax. 8th.

Suppoſe A to B, and C to D, &c. are incommenſurable, but in the ſame Ratio; ſtill, A+C+E:B+D+F, &c. ::A:B, &c. becauſe, A might either be equal to C, or E, or it might be ſome Multiple or equal Part of C, and C of E, &c.

Or, ſuppoſe A be incommenſurable to C, and C to E, &c. provided A and B, C and D, &c. be commenſurable, the ſame thing may be done; as in the Theorem.

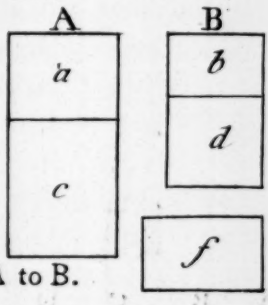
But, being incommenſurable both ways, there cannot poſſibly be equality in either; and the Demonſtration given (or rather preſumed) from a greater or leſs Ratio, is by no means poſitive; ſeeing, it is not aſcertained in what Degree they are ſo.

For, A may have to B, and C to D, &c. the Ratio of the Side of a Square to its Diagonal, or as the Diagonal to the Side, &c. and A to C may be the ſame; or, as the Diameter of a Circle to the Circumference; and C to E, in extreme and mean Proportion.

Yet it is eaſy to conceive, from what has been proved, that (let the Ratio of A to B be what it may) ſince the Ratio of C to D, and of E to F, &c. is the ſame, conſequently, A+C:B+D, or A+C+E:B+D+F::A:B, or C to D, &c. and although it is not poſſible to give direct proof, in ſuch Caſe, yet we may ſafely conclude that it is ſo; and, on that preſumption, all which follows will be found poſitive, let the Ratio be what it may.

THEOREM III. 19 Euclid.

In two Quantities, if a Part taken from one, be in the ſame Ratio to a Part taken from the other, as one Quantity is to the other; the remainders of thoſe Quantities are alſo in the ſame Ratio, as one Quantity is to the other.



Let A and B be two Quantities, in any Ratio whatever, and, ſuppoſe any part of A be taken (as *a*) from A.

Let *b* be taken, from B, in the ſame proportion to a, as A to B.

I ſay, the remaining Part *c*, is to *d*, as A to B.

Let any other Quantity (*f*) be ſo taken, *a*:*b*::*e*:*f*. Poſt 3.

Dem. Now, *a*:*b*::*c*:*f*; and *a*:*b*::A:B; conſ. *c*:*f*::A:B.-Ax.13
But, A:B::*a*+*c*:*b*+*d* (Ax.1.) and, as *a*:*b*:*c*:*f*. - Poſt.
wherefore, *a*+*c*:*b*+*f*::*a*:*b*; i. e. as A:B. - Th. 2
conſ. *a*+*c*:*b*+*f*::*a*+*c*:*b*+*d*; wh. *b*+*f*=*b*+*d*. - Ax.5
And by taking away *b*, which is common, *f*=*d*.-Ax.7.1
But, *c*:*f*::*a*:*b* (Poſt.) conſequently, *c*:*d*::*a*:*b*. - Ax. 4
And, *a*:*b*::A:B (Hyp.) Therefore, *c*:*d*::A:B. - Ax.13

Cor. If one Quantity be to another as a third is to a fourth; and, if a fifth be to the ſixth, as the third to the fourth; then, the firſt and fifth, added together, has the ſame Ratio to the ſecond and ſixth, together, as the third has to the fourth.

THEOREM IV. 16 Euclid.

If four Quantities, of the ſame kind, are Proportionals; i. e. taken two and two, in a certain order, they are directly proportional, or analogous in their Ratio; they are alſo analogous when taken alternately.

If A is to B, as C is to D; then, A is to C, as B to D.

Dem. Now, ſince A : B :: C : D; whether A be equal to, greater or leſs than B; C is alſo equal to, greater or leſs than D. - - - - - - - Ax. 11.

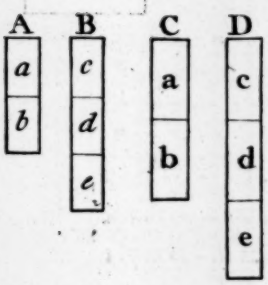
And, if A be equal to, greater, or leſs than C; B is alſo equal to, greater, or leſs than D. - - - 12.

But, A has the ſame Ratio to B as C to D; - - Hyp.

it follows, then, that A has the ſame to C, as B to D.

For, whether A and B be equimultiples, or equal parts of C and D; A is in the ſame Ratio to C, as B to D - Ax. 7.

ſeeing that A is to B as C to D. - - - - - - 8.



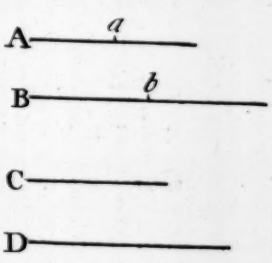
Suppoſe the Ratio of A to B be as 2 to 3; and, imagine them divided into equal Parts, in *a* and *b*; *c*, *d*, and *e*.

Alſo, ſuppoſe C and D divided into the ſame number of equal Parts, reſpectively, in a and b; c, d, and e.

Then

Then, since A:B::2:3, & A:B::C:D; C:D::2:3-Ax.13
wherefore, the Parts, a and b, (of C) and c, d, e, (of D)
are all equal amongſt themſelves. - - Ax. 9.
Conſequently, as *a*:a, or *b*:b, ::*c*:c, *d*:d, or *e*:e. - Ax. 4.
But, *a* and *b* are equal parts of A; and, a and b, of C.
Alſo, *c*, *d*, & *e*. are equal parts of B; and, c, d, and e, of D.
But, Quantities are, to each other, in the ſame Ratio as
their equimultiples, or equal parts. - - - Ax. 8.
Therefore, as A is to C, ſo is B to D. Q. E. D.

Again. Suppoſe the Ratio of A to B, and of C to D, be incommenſurable; and conſequently, cannot be divided into Parts, which are all equal amongſt themſelves;



Then, let A and B be divided into the ſame number of equal Parts, ſuppoſe they are biſected, in *a* and *b*. - - Poſt. 2

Now, A:B::C:D, and A:B::*a*:*b*, - - - - Ax. 8.
conſequently, C:D::*a*:*b*. - - - - - Ax. 13.
But, A:*a*::B:*b* (Ax. 7) and, as *a*:*b*::C:D. - as above.
Wherefore, by ſubſtituting one for the other, - Poſt. 1.
conſequently, as A is to C, ſo is B to D. Q. E. D.

THEOREM V.

Quantities, which are analogous when taken directly, are alſo analogous, taken inverſely.

If A is to B, as C is to D; then, B is to A, as D to C.

Dem. For, if A:B::C:D; then, A:C::B:D, - - by 4.
i. e. B:D::A:C; then, B:A::D:C. - - ſame.

THE-

THEOREM VI. 18 Euclid.

Quantities which are analogous, taken directly, are alſo analogous, when compounded.

If A is to B as C is to D.
Then A added to B, is to A or B; as C added to D, is to C or D.

DEM. For, ſince A:B::C:D; then, A:C::B:D - Th. 4.
wh. as A:C::A+B:C+D. (2) conſ. A:A+B::C:C+D.
and conſequently, as A+B:A::C+D:C. - - Th. 5.
But, B:D::A:C; therefore, A+B:B::C+D:D - Ax. 13.

COR. By inverſion, A, or B, :A+B::C, or D, :C+D.

THEOREM VII. 17 Euclid.

Quantities which are analogous, directly, are alſo analogous when divided.

If, in four Quantities, A, B, C, and D; A is to B, as C to D;
Then, according as A or B is the greater Quantity,
A leſs B (or B leſs A) is to A, or B,
as C leſs D (or D leſs C) is to C, or D.

DEM. Now, A:B::C:D. (Hyp.) wherefore, A:C::B:D - 4.
But, as A:C, (or B to D)::A−B (or B−A):C−D - 3.
conſ. A (or B) :A−B::C (or D) :C−D - - - 4.
and, conſ. A−B :A (or B) ::C−D:C (or D) - - 5.
Or; - - B−A:A, or B, ::D−C:C, or D.

COR. The Converſe of this is firſt proved, in the third ſtep.

THE-

THEOREM VIII.

If four Quantities are Proportionals, in a certain order, they are alſo proportional when mixed; i.e. their Sums and Differences are analogous.

If A is to B as C to D.

Then, A+B:A−B (or B−A) ::C+D:C−D (or D−C)

Dem. Now, A:B::C:D (Hyp.) then A:C::B:D - Th. 4.
Wherefore, as A:C::A+B:C+D - - - - 2.
alſo, as A:C::A−B:C−D. - - - - 3.
conſequently, A+B:C+D::A−B:C−D - - Ax. 13.
therefore, A+B:A−B::C+D:C−D. - Th. 4.

Cor. A−B:A+B::C−D:C+D, by inverſion. - 5.

THEOREM IX. 22 Euclid.

If there be three or more Quantities, and others equal to them in number; which, taken two and two in a certain order (ordinately, or progreſſively) are in the ſame Ratio; then will the firſt be to the laſt, as the firſt to the laſt, in each Order of Quantities.

Alſo, as the firſt is to the firſt, ſo is the laſt to the laſt.

If A:B::C:D; and, if B:E::D:F; A is to E as C to F.

Dem. Now, A:B::C:D; wherefore, A:C::B:D } - Th. 4.
And - B:E::D:F; wherefore, B:D::E:F }
Now, ſince A:C::B:D; and, as B:D::E:F.
conſ. A:C::E:F (Ax. 13.) Therefore, A:E::C:F. - 4.

N. B. The laſt Part is proved firſt.

Again. If other Quantities, G and H, be added, in the ſame Ratio to the laſt Conſequents, reſpectively; as E is to G, ſo is F to H; it is manifeſt, that A is to G, as C to H; the firſt, ſtill, in the ſame Ratio to the laſt.

For, as A:E::C:F; and, as E:G::F:H; by Theorem. wherefore, A:C::E:F; and, as E:F::G:H; by the 4th. Conſ. as A:C::G:H (Ax. 13) th. A:G::C:H - ſame.

And thus it will ever be, if Quantities be added (in equal Ratio) after the ſame Order, ad infinitum.

COR. Hence, if the Quantities, added are in equal Ratio to the Antecedents, reſpectively;
thus, if A:B::C:D, and, if E be to A as F to C;
it is manifeſt, that as E is to B, ſo is F to D, &c.

THEOREM X. 23 Euclid.

In two Ranks or order of Quantities, if there be three or more in each; which, taken two and two, inordinately, are in the ſame Ratio; then, as the firſt is to the laſt, in one Order, ſo is the firſt to the laſt in the other; as by equality.

Let A, B, C, and D, E, F, be Quantities in ſuch Order, ſo that, as A is to B, ſo is E to F; and, as B is to C, ſo is D to E.

I ſay, that, as A is to C, ſo is D to F.

Let any other Quantity, G, be taken ſo; that, as D is to E, ſo is F to G; - Poſt. 3.
conſ. as D is to F, ſo is E to G. - - 4.

DEM.

Dem. Then, as before, A:B::E:F, wh. A:E::B:F. - 4.
Alſo, as B:C::D:E, i. e. as F:G, wh B:F::C:G-ſame.
Now, A:E::B:F; and B:F::C:G; wh. A:E:C:G-Ax.13.
and conſequently, as A:C::E:G. - - - - - Th. 4.
But, D:F::E:G, (above) Th. as A:C::D:F - Ax.13.

If other Quantities be added, inordinately, in equal Ratio, it will ſtill be as the firſt is to the laſt, in one Rank, ſo is the firſt to the laſt, in the other. e. g.

Let the Quantities, G and H, be ſo taken, that, as C is to G, ſo is H to D; then, as A is G, ſo is H to F.

For, as A:C::D:F (above) and, as C:G::H:D - Hyp.
conſequently, as A is to G, ſo is H to F; - - as before.

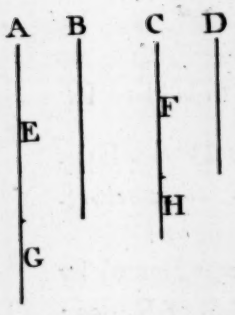
Inordinate equality may be proved, (after ordinate) by means of the 21ſt Definition, viz. by compoſition of Ratios; but, as that Definition is ſomewhat obſcure, the Demonſtration ariſing, therefrom, would be ſo too.

For, ſince A:B::E:F; and, as B:C::D:E; alſo as C:G::H:D; it is manifeſt, ſeeing that there is equal Ratios between the two extremes, A and G, H and F; let the intermediate Ratios be ever ſo many, and in what order ſoever, provided there be an equal number in each Rank; and the extreme Ratios are equal, ordinately or inordinately; alſo, that there be found equal Ratios between the Extremes, in each Rank; two and two, any how ſituated; the Ratio, compounded of them all, is, as the firſt to the laſt, in each Rank, or order of Quantities; by the Definition.

But, the extreme Ratios are equal (ordinately or inordinately) and conſequently, the extreme Quantities, of both Ranks, are in equal Ratio.

THEOREM XI. 25 Euclid.

If four Quantities are Proportionals, in a certain order; the greateſt and the leaſt are, together, greater than the other two.



Let A, B, C, and D, be four Quantities, of which, let A be the greateſt; and let them be, as A to B, ſo is C to D; conſequently, D, is the leaſt Quantity.

For, if the firſt be greater than the ſecond, the third is greater than the fourth - Ax. 11. and, if the firſt be greater than the third, the ſecond is greater than the fourth. - 12.

DEM. Becauſe A and C are greater Quantities than B & D; let E and F be taken from A and B, reſpectively, equal to B and D, reſpectively.

Now, A:B::C:D. But E=B, and F=D - Con. wherefore, as A:E::C:F. - - - Ax. 4. conſ. A—E (i. e. G) :A::C—F (i. e. H) :C - Th. 7. i. e. G:A::H:C; wherefore, G : H :: A : C - Th. 4. But, A is greater than C; wh. G is greater than H - Ax. 11. conſ. G+E (i. e. A) +D, is greater than H+F (i.e. C) +B. Th. A+D is greater than C+B, the two middle Terms.

In the laſt ten Theorems, and the Corollaries deducible from them, is contained all the various ways of reaſoning, by Proportion, which I conceive to be uſeful; two of which, the fifth and eighth, are not proved by Euclid. The eighth, indeed, is not mentioned, by him, or any of his Followers, that I am acquainted with; but, I am ſomewhat ſurprized, that none

none have given a Demonſtration of Inverſe Ratio; which being proved, every Converſe Ratio is alſo proved. Profeſſor Simſon is aware of that deficiency, and has very judiciouſly introduced it, as an additional Propoſition (B;) after which, I muſt think it needleſs, to add the Converſe of divided Ratio (E) unleſs he had, alſo, given the Converſe of all.

But what, to me, ſeems very extraordinary is, that Euclid ſhould not demonſtrate either; alſo, that he ſhould prove (if it may be ſaid to be proved) divided Ratio (the 17th) before compounded (the 18th) when the latter is, in ſome meaſure, made a Condition of the former; they are, indeed, Converſe, each of the other. In the 17th he ſays, "if Magnitudes, taken jointly or compounded, be proportional, they are alſo proportional taken ſeparately, or divided." Now, by the tenor of theſe Premiſes, *if* one is, the other is, alſo; and conſequently, *if* one is not, the other is not; whereas, the *Condition* is preſumed, only; and conſequently, there is no proof of either. But where is the neceſſity of proving it on that Condition, when it may be done as elegantly, ſolidly, and briefly without it, from the ſimple Analogy of the two Ratios.

Either I am not clear, in the nature and meaning of compounded and divided Ratio, or thoſe Propoſitions are not to the purpoſe.

By Definitions 15th and 16th is to be underſtood, when Quantities are proportional (ſimply) they are alſo proportional compounded or divided; which, Profeſſor Simſon has expreſsly ſaid, in his Definitions of them; and which, I have demonſtrated in the 6th and 7th Theorems. Now, his 17th Propoſition ſays, as above; and *vice verſa* in the 18th, viz. that, if Quantities are Proportionals, when compounded or divided, they are ſo, divided or compounded; which, I cannot conceive to be the true meaning, but as I have ſtated them, in the Theorems, and have demonſtrated, on that Hypotheſis.

If what I have advanced, on the ſublime Doctrine of Proportion, be not ſufficient, for any purpoſe whatever, I will be bold to ſay, that it is not ſo in Euclid, or any of his Commentators. The Axioms, which I have given, are gathered from the moſt eaſy and ſimple ideas of Proportion; after the Definitions are digeſted and clearly underſtood, there is no Perſon would heſitate, one moment, to grant every Axiom; they require no proof, being ſelf evident.

That Quantities are in the ſame Ratio as their Equimultiples, or equal Parts (the 8th) is manifeſt to the meaneſt capacity; and this is one of the principal, on which Euclid has founded his whole Theory of Proportion; which amounts to no more than this; that one is to two, three, &c. as, two, is to four, ſix, &c; or, as three to ſix, nine, &c. i. e. as one, to two, three, &c. Or, that one Quantity is to any other Quantity, as the half, or third part, &c. of one, is to the half, or third part, &c. of the other.

The 1ſt Axiom, in the firſt Book of Elements, of Euclid, (the 3rd of theſe Elements) expreſly ſays "*Things*, which are equal to the ſame *Thing*, are equal between themſelves." Now, the queſtion is, whether *Ratios* are *Things?* Certainly, if any thing be meant by Ratios, they muſt come within that Appellation; for, granting the Term to be, in itſelf, metaphyſical; yet, *ſomething* is underſtood and meant by Ratio, and conſequently, *Ratios* are *Things*. Then, the 11th Propoſition of Euclid's fifth Book, is as much an Axiom as the firſt of the firſt; and, I have accordingly made it ſo, in the thirteenth. The remainder of the Axioms are as ſimple, obvious, and ſelf evident; which, with the Poſtulates, being granted, the whole of the Theorems, I am confident, will be found ſolidly, yet briefly, and clearly demonſtrated.

A critical

A critical Remark on Euclid's fifth Definition, of the fifth Book.

Euclid in his fifth Definition, has given us a Criterion, whereby to determine Analogy of Ratios, rather than a ſimple Definition of what it is.

He ſays, "Magnitudes (by which is meant Quantities of any kind) are ſaid to be in the ſame Proportion, the firſt to the ſecond, and the third to the fourth, when the equimultiples of the firſt and third, being compared with the equimultiples of the ſecond and fourth, according to any multiplication; either together exceed the one the other, or together are equal the one to the other, or together are leſs, the one than the other." This is, litterally, according to the old Tranſlation by H. Billinſley in 1570.

Dr. Barrow ſays, "Magnitudes are in the ſame Ratio, when the Equimultiples of the firſt and third, compared with the equimultiples of the ſecond and fourth, according to any multiplication whatſoever, either both together are leſs than the ſecond and fourth both together, or equal taken together, or exceed one the other together, if thoſe be taken which anſwer one to the other."

Dr. Keil, from the latin tranſlation of Commandine, has nearly the ſame words, viz. "the Equimultiples of the firſt and third (compared as above) are either both together greater, equal, or leſs than the equimultiples of the ſecond and fourth; if thoſe be taken that anſwer each other."

Cunn did well to explain the meaning of Dr. Keil's Definition, which is literally this; that, equimultiples of the firſt and third, taken or added together (i. e. into one Sum or Quantity) are either equal, greater, or leſs than the equimultiples of the ſecond and fourth. I think, there is great occaſion for a comment on Dr. Barrow's; which, to me, is quite unintelligible.

After all that has been ſaid, concerning this Definition, by the learned Doctor and others, either the original Text muſt be very obſcure, or the expounders have rendered it ſo, for I declare, I cannot think the words bear ſenſe. But, giving the greateſt latitude poſſible to the meaning of the words, they imply, the firſt and the third taken together, and compared with the ſecond and fourth, together, (i. e. in one Sum) in which Caſe it amounts to a clear Axiom; for they muſt, certainly, be either equal, greater, or leſs, one than the other; after which, the words, "if thoſe be taken which anſwer the one to the other," mean nothing at all.

Mr.

Mr. Stone ſays, "they are, together, either deficient, equal or exceed *each* other." They may, indeed, be equal, *each* to the other; but, it is impoſſible that they can, at the ſame time, be either greater or leſs, *each* than the other.

The word, *together*, ſeems, to me, quite ſuperfluous; or elſe the tenor of it is, in this place, *at the ſame time*, or, *by the ſame multiplication*; which is quite foreign to the common acceptation of it. It is, perhaps, better to omit it entirely, and ſupply its place with *both* (which Barrow and Keil have added) it is then plain; the firſt and the third, compared with the ſecond and fourth, will be either *both* greater, *both* equal or *both* leſs; which is the concluſion all have drawn from it; and then, the words, if thoſe be taken which anſwer each other, ſeem neceſſary.

Mr. Stone in vindicating and illuſtrating this Definition, ſhews, that it is clearly deduced from the fourth and fourteenth Propoſitions. According to the Idea I have of the Definitions of Terms, in any Science, they ought to be defined by their moſt natural and ſimple properties; from which the demonſtrations of ſome Theorems are to be obtained; for it ſeems inconſiſtent, firſt to demonſtrate ſeveral preceding Propoſitions, by means of a certain Definition, previous to the demonſtration of others, from which the Definition, itſelf, is to be deduced, and demonſtrated.

It is true, he ſays "it is not ſo ſimple and plain as the Definition of Numbers, or that which might be given of commenſurable Magnitudes." And again: "Euclid could not have given any other, ſo elegant and general a Definition, that would have taken in incommenſurable Magnitudes, as well as Numbers, and commenſurable ones.

It is well known, that the Diagonal of a Square is not commenſurable to its Side; alſo, the Circumference of a Circle is incommenſurable by its Diameter or Radius; that is, their Ratios cannot be expreſſed in Numbers; for, the Side of a Square cannot be divided into any number of parts, though ever ſo ſmall, which will be an aliquot part of the Diagonal, i. e. they cannot both be divided into the ſame equal parts, or into parts of equal magnitude, without defect; ſuch Quantities are, therefore, incommenſurable.

The Ratio, between the Side of a Square, and its Diagonal is very different from that, between the Diameter of a Circle and its Circumference. Yet, the Ratio between the Side of every Square and its Diagonal, or between the Diameter of every Circle and its Circumference, is the ſame; i. e. their Ratios are equal or analogous.

It is intimated above, and is ſtrongly ſupported by Dr. Barrow and others, that this Definition comprehends or extends to Incommenſurables

commenſurables. I am, however, perſuaded, that it is no eaſy matter to determine the analogy of ſuch Ratios by it. The rule it preſcribes, or is intended to preſcribe, is ſimply this. Take any Equimultiples of the firſt and third Terms, in four Quantities, that are preſumed to be analogous in their Ratios; i. e. multiply the two Antecedent Terms, equally, the ſame number of Times, and, either by the ſame Number or any other, multiply the two Conſequents, the third and fourth, or take Equimultiples of them; then, the firſt and the third, being compared with the ſecond and fourth; if the firſt exceeds the ſecond, the third will alſo exceed the fourth, if one be equal, the other is alſo equal, or they will be both leſs, than the ſecond and fourth; i. e. the two Antecedent Terms will, at the ſame time, be either both equal, both greater, or both leſs than their reſpective Conſequents; and, when this is the caſe, in all multiplications, *whatever*, we may then conclude that their Ratios are analogous.

Now, if the Ratios are of incommenſurable Quantities, which cannot be expreſſed by Numbers, thoſe Quantities cannot be multiplied by Numbers; and, in Lines, it would require greater accuracy than it is poſſible for any perſon to apply, to take Equimultiples of them. It is certain, that being multiplied, mentally, according to this Definition, if they were analogous, the Antecedents would be either greater or leſs than the Conſequents, continually, but never equal; for if, by taking Equimultiples, they could be equal, they may, conſequently, alſo be divided into equal Parts; but, it is well known, they cannot; which, negative Quality, is the diſtinguiſhing characteriſtic of Incommenſurables. And, ſince it is manifeſt that the Antecedents of Ratios which are not analogous may, by multiplying, be either greater or leſs than their reſpective Conſequents, but never both equal, at the ſame time, i. e. by the ſame multiplication; how, then, is it poſſible to diſcover analogy of Ratios by this Criterion? ſeeing that, in Ratios of commenſurable Quantities, which are not analogous, they may be greater or leſs, but never equal, as they will ever be in incommenſurable Quantities that are analogous.

Hence, it appears, to me, that, to produce equality, of the Antecedent to the Conſequent, is the only certain and infallible ſign of Proportionality, or analogous Ratio, between commenſurable Quantities; but, ſince that Criterion fails, in Incommenſurables, there cannot, by its means, be any poſitive determination concerning Analogy, in ſuch Quantities; unleſs, by trying *all multiplications*, *whatever*, they are, *at all times*, either *both* greater, or *both* leſs.

If the Antecedents are either both greater, equal, or leſs than their reſpective Conſequents (and no Perſon would look for Proportionality otherwiſe) it is manifeſt, that, by multiplying the

Antecedents any number of times, equally, and the Conſequents the ſame, or any other, alſo equally; they muſt neceſſarily be, at the ſame time, either both greater or both leſs; but, if they are not analogous, they never can, at the ſame time, be both equal, i. e. each Antecedent to its reſpective Conſequent; and therefore, no other Sign, but equality, is neceſſary, or ſufficient, to prove Analogy of Ratios.

I am perſuaded, that either Euclid, himſelf, or his Advocates are under ſome miſtake in reſpect of this fifth Definition; which does not, poſitively, determine the Analogy of incommenſurable Quantities. Therefore, the inveſtigation, by means of it, is vague and imperfect; and conſequently, all that is built, entirely on it, is ſo too; ſeeing it cannot, as it is preſumed, diſcover Analogy, in incommenſurable Quantities; becauſe, equality of the Antecedent to the Conſequent (the only certain Sign) cannot, by multiplying, be produced.

It is manifeſt, that if Equimultiples of incommenſurable Quantities are taken, they will ſtill be in the ſame Ratio; i. e. if the ſide of a Square and its Diagonal, &c. be equally multiplied; but no multiples of the Side and of the Diagonal can be ſo taken, that they may be equal to one another; for, it is evident, if they could, they might, alſo (as obſerved above) be divided into the ſame equal Parts, which cannot be. And, for ſuch as are commenſurable, it is an unneceſſary and unſatisfactory method of determining.

Then, ſince no ſign, but equality of the Antecedent to the Conſequent, can be poſitive, and ſince that cannot ſubſiſt between Incommenſurables, although it is certain, that *if* one is, the other muſt neceſſarily be ſo too; but ſince that, *if*, can never poſſibly happen, we cannot, on that ſuppoſition, determine Analogy of Ratios, in ſuch Quantities.

The Reverend Dr. Barrow, in his 21ſt Lecture, has made a very learned Defence of this Definition; and, ſpiritedly, encounters the Detractors of Euclid, viz. Ramus, Tacquet, and Hobbs, in the 22nd. But, having ſilenced thoſe weak Opponents, he is cloſe put to it by Borellus, in his 23rd and laſt Lecture; where, after various ſkirmiſhes, in which, the Doctor has not always the advantage, he briefly ſums up the whole Evidence, in theſe words, "that, in his judgment, there is nothing extant in the whole work of the Elements, more ſubtilly invented, more ſolidly eſtabliſhed, or more accurately handled, than the Doctrine of Proportionalities." Nevertheleſs, if his own Definition of it is to be the Touchſtone, I cannot ſubſcribe to his Atteſtation.

Now, as I have before obſerved, in the Preamble to this fifth Book, Proportion is, in a great meaſure, an innate principle; which, being clearly explained, what is meant by it, and explicated by Numbers,

bers, is ſo very intelligible, that, to any tolerable Capacity, it does not require other Demonſtration; and all that is built on that knowledge is, I am confident, as ſolid and permanent, and as ſecurely eſtabliſhed, as by all the Demonſtration which can poſſibly be given; at leaſt, on the foundation of Euclid's fifth Definition.

In reſpect of commenſurable Quantities, to all which Numbers may be applied, whether their Ratios are analogous or not, is readily diſcovered; ſince, in Analogy, the Ratios being equal, the ſame Numbers will expreſs either.

E. g. 56:98::12:21; which being reduced to their loweſt Denomination, the Ratio, of each, is determined. The firſt pair, 56 and 98, being divided, ſeparately, by 14, gives 4 and 7, the true Ratio of that pair. Now, if the other pair produces the ſame, they are analogous; 12 and 21, divided ſingly, by 3, produce, alſo, 4 and 7; which is, therefore, the true Ratio of both; conſequently they are analogous, ſeeing the Ratios are equal.

If there be taken Equimultiples of Ratios that are analogous, i. e. being multiplied various ways, either all by one Number, (any number whatever) or the firſt pair by one Number, and the ſecond pair by another; or, if the firſt and third be multiplied by one Number, and the ſecond and fourth by any other, the Products ariſing, from every ſuch multiplication, are ſtill analogous, in the Ratio of both Pairs.

Let A : B :: C : D, in the Ratio of 3 to 5.
Thus, 9 : 15 :: 3 : 5.

So, 27 : 45 :: 9 : 15, being all multiplied by 3.
Alſo, 18 : 30 :: 15 : 25; the firſt Pair by 2, and the ſecond by 5.

In both theſe Caſes, the Ratio is ſtill as A to B, or as C to D.

Again, 36:30::12:10, { the firſt and third, being multiplied by 4, are both greater than the ſecond and fourth, multiplied by 2.

Alſo, 45:45::15:15, { the firſt and third multiplied by 5, the ſecond and fourth by 3; in which caſe they are both equal.

And, being all multiplied by any one Number, as above, it is evident they will both be leſs. Or, being multiplyed by various Numbers, they will both be leſs; which illuſtrates the fifth Definition of Euclid.

Now, it is very obvious, in Numbers, from any one, of theſe various operations, that there is analogy of Ratios between A and B, C and D; and it is full as obvious without. But, provided it is not known, whether the Ratios are analogous or not, no one, nor all of the three firſt trials are ſufficient to determine it, merely from the firſt and third being, at the ſame time, both greater or both leſs than the ſecond and fourth; becauſe, thoſe things may, and very frequently do, happen, where there is not Analogy.

E. g. Let the Ratio of A to B be as 3 to 4; and, C:D::$2\frac{3}{4}$:4.
In whole Numbers, A:B::3:4, or 12:16, and C:D::11:16.

Now, let there be taken Equimultiples of A and C, and other equimultiples of B and D. e. g. Let the Antecedents be taken thrice, and the Conſequents twice; it will be 3A : 2B :: 9:8, and 3C : 2D :: 33 : 32; by which multiplication, it is evident that the Antecedents are both greater than their Conſequents.

But, ſeeing that the Ratios are not equal, it is impoſſible to take Equimultiples of the two Antecedents, and alſo Equimultiples of the Conſequents, by which, the Products of the Antecedents ſhall be both equal, reſpectively, to their Conſequents; and alſo, becauſe their Ratios are not equal, ſuch Equimultiples may be taken of them, that one Antecedent ſhall be greater, and the other leſs, than its reſpective Conſequent.

E. g. Let A and C be taken four times, and B and D thrice, it will then be, 4A:3B::12:12, and 4C:3D::44:48;

Again; let A and C be taken 7 times, and B and D 5 times; it will be 7A : 5B :: 21 : 20; and, 7C : 5D :: 77 : 80.

Either of theſe operations is ſufficient to evince, that the Ratios are not analogous; i. e. A is not to B, as C to D; for, it is impoſſible that equality of Antecedent to Conſequent ſhould happen in both pairs, (i. e. of A to B, and of C to D), unleſs they are analogous; which is the only indiſputable proof of Analogy; for, one Quantity muſt neceſſarily be to itſelf, as any other Quantity is to itſelf. And ſince it is not poſſible for equality to happen, by multiplication any more than by diviſion, between incommenſurable Quantities, it is manifeſt, that Analogy of ſuch Ratios cannot be abſolutely determined by that Criterion.

Wherefore, the Demonſtration, according to Euclid, in the fifth Book is not poſitive, in reſpect either of commenſurable or incommenſurable Quantities; but is only preſumptive, at beſt, and is very obſcure and unſatisfactory; ſince it is manifeſt, that, in unequal Ratios, Equimultiples may be ſo taken (according to Euclid) in which the Antecedents will be either both greater or both leſs than their Conſequents (as it has been ſhewn above) but, it is not poſſible they can be taken, ſo, that both Antecedents ſhall be equal to their Conſequents.

I am therefore of Opinion, that, the Doctrine of Proportion is not properly inveſtigated by Equimultiples; nor can I conceive, that taking multiples of Quantities, in order to prove the equality of their Ratios, is properly geometrical. It ſeems, to me, more conſiſtent to compare Quantities by themſelves, either whole and entire, or by their equal Parts.

But it is manifeſt, that there are Ratios, which will not admit of diviſion into the ſame Parts, i. e. into Parts of the ſame magnitude; yet it is certain, that they may, with equal eaſe and more propriety, be divided into the ſame number of Parts, as to take Equimultiples of them; which Method I have therefore adopted, as the moſt rational and eligible.

ELEMENTS OF GEOMETRY.

BOOK VI.

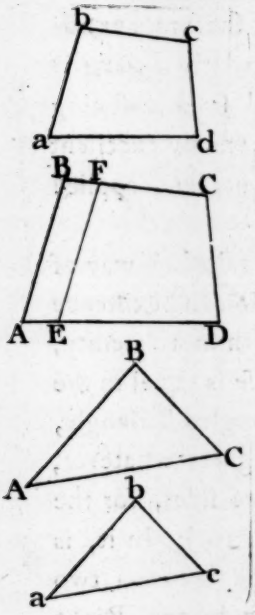
THE ſixth Book of Elements is not only the moſt uſeful and valuable of the whole, but alſo the moſt entertaining and inſtructive. In it, the Doctrine of Proportion is applied to real uſe, in finding out the more extraordinary properties of Plane Figures, (particularly Triangles and Parallelograms) by the moſt ſubtle and ſolid reaſoning that can be conceived: by means of which, many excellent and extenſively uſeful Problems are found out, and applied to various and moſt notable purpoſes.

In it, the Rule of Three, or Proportion, called by way of eminence, the Golden Rule, has its foundation and exiſtence (Theo. 9.) in it, is alſo demonſtrated, with more facility, that, not only the Square of the Hypothenuſe is equal to the two Squares of the Sides, of every right angled Triangle, but likewiſe, that it is ſo, in all ſimilar Figures whatever, regular or irregular, conſtructed on the three ſides, for the correſponding ones of each Figure. (Theo. 16.) In it, is determined the juſt proportion which exiſts between two ſimilar Figures of any kind, and exhibited by two Right Lines (12 and 13) by which means, any Figure, whatever, may be conſtructed, in any Ratio to a given one, (Prob. 37 and 16;) or, the Side of a Square may be determined, whoſe Area ſhall be equal to that of any given right lined Figure, whatever. (Prob. 24 and 25.) And, what is more extraordinary, a right lined Figure may be conſtructed ſimilar to any given Figure, and equal to any other, of different Forms and Denominations, being right lined. (Prob. 38.)

In it, ſeveral valuable properties of the Circle are briefly demonſtrated; in ſhort, to ſee and admire all which it contains, I recommend the careful and attentive peruſal of it, as the readieſt and moſt certain means, by which its beauties and excellencies can poſſibly be communicated.

DEFINITIONS.

Def. I. SIMILAR FIGURES are ſuch as have all their Angles equal, each to the other, reſpectively; and alſo, the Sides which are oppoſite to equal Angles, or which lie between equal Angles are proportional.



The Quadrilateral a b c d, having all its Angles equal, reſpectively, to the Angles of the Quadrilateral ABCD, viz. a equal A, b equal B, &c; and, if, the Side a b have the ſame Ratio to AB, as b c has to BC, &c. then, a b c d is ſimilar, or like, ABCD.

If EF be drawn parallel to A B, the Angles at E and F, are equal to A and B, reſpectively (4. 1.) conſequently to a and b; wherefore, all the Angles of the Quadrilateral a b c d, are equal to thoſe of EFCD; but, ſeeing the correſponding Sides, a b and E F, b c and FC, &c. are not proportional with c d and CD, a b c d is not ſimilar to E F C D.

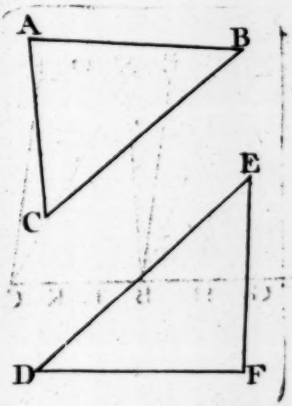
Triangles ABC, a b c, havingall the Angles of one, equal, reſpectively, to the Angles of the other, have their Sides neceſſarily proportional, and are, conſequently, ſimilar Figures.

The Sides a b, A B; b c, BC, &c. being oppoſite to equal Angles, or which lie between equal Angles, are HOMOLOGOUS or correſponding Sides.

N. B. All ordinate Figures, whatever, i. e. ſuch as are equilateral and equiangular, one with another, are ſimilar Figures; as equilateral Triangles, Squares, &c. Alſo all Circles are ſimilar Figures.

Def. II. In Triangles, and Parallelograms, if there be two Sides, in one, which, with two Sides in another, of the ſame kind, and about the ſame Angle, are Proportionals; in ſuch wiſe, that the Antecedent of one, and the Conſequent of the other pair, are in the ſame Figure, thoſe Figures are ſaid to be RECIPROCAL.

But more properly, the Ratio is reciprocal in ſuch Figures; for, reciprocal Figures is unmeaning.



In the Triangles ABC, DEF; if the Sides AB and BC of the one, and DE, DF, of the other, are Proportionals; and, if AB is to DE, as DF, of the ſame Triangle, is to BC of the other; then, the Proportionals are reciprocal in thoſe Figures; and they are uſually, though improperly, called reciprocal Figures.

Underſtand the ſame of Parallelograms.

N. B. Equiangular, or ſimilar, Triangles are not reciprocal; becauſe their Sides are all, reſpectively, proportional; which is not the Caſe, in the Figures annexed. For, EF is not neceſſarily in the ſame Ratio to AC, as are the other Sides, AB to DE, and DF to BC. Neither can there, in ſimilar Figures, be found the Antecedent of one Pair, and the Conſequent of the other, in the ſame Figure, unleſs they are congruous. (See Def. 43.)

The Sides of ſimilar Figures are, therefore, directly proportional, and not reciprocally.

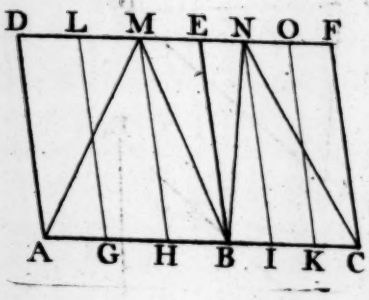
For the Definitions of BASE and ALTITUDE, ſee Def. 45 and 46 in the general Introduction.

AXIOM. If two, or more, Figures are ſimilar to the ſame Figure, they are ſimilar between themſelves.

THE-

THEOREM I. 1 Euclid.

Parallelograms, or Triangles, having the ſame Altitude, are, to each other, in the ſame Ratio as their Baſes.



Let ADEB and BEFC be two Parallelograms, between the ſame Parallels, AC and DF.

I ſay, the Ratio of the Parallelogram AE to BF, (i. e. of their Areas) is, as AB to BC, their Baſes.

And, the Triangle AMB is to BNC, as the Baſe, AB is to BC.

Let AB be divided into any number of Parts, equal to each other, in G and H; and ſuppoſe BC divided into the ſame number of equal parts, in I and K (Poſt. 2. 5.)

Draw, GL, HM, &c. parallel to AD and BE.

DEM. Becauſe AG, GH, and HB are equal to each other; the Parallelograms AL, GM, and MB, are equal. - 18.1.

And, the Pars. BN, NK, and KF, are alſo equal - ſame

Now, if AB = BC, the parts AG &c. are equal to BI, &c.

If AB be greater than BC, the part AG is greater than BI; and, if AB be leſs than BC, AG is leſs than BI.

Conſequently, the Parallelograms AL, GM &c. are, alſo, either equal to, greater, or leſs, than BN, NK, &c. as, AG, &c. is equal to, greater or leſs than BI, &c.

For, Quantities are in the ſame Ratio, to each other, as their Equimultiples, or equal Parts. - - Ax. 8.5.

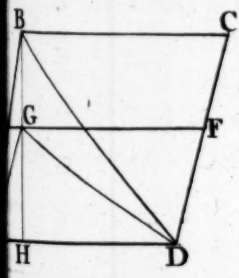
Now,

Now, AG:AB::BI:BC
And, Par. AL:AE::BN:BF } in equal Ratio,-Ax. 7. 5.
Conſ. Par. AL:BN::AG:BI, i.e.::AB:BC,(P.1.5.)-4.5.
But, Par. AL:BN::AE:BF. - - - ſame.
Th. Par. AE:BF::AB:BC. - - Ax. 13. 5.

2nd. Becauſe Triangles are equal to half Parallelograms, having the ſame Baſe and Altitude; - - 17. 1.
the Parallelogram AE:BF::△.AMB:BNC - Ax. 8. 5.
and, Parallelogram, AE:BF::AB:BC - above;
conſ. the Triangle AMB:BNC::AB:BC. - 13. 5.

Cor. Parallelograms or Triangles having the ſame or equal Baſes, have the ſame Ratio to each other, as their Altitudes.

For, if their Baſes be conſidered as their Altitudes, and their Altitudes as their Baſes, it is evident from the Theorem.



Let ABCD and AEFD be two Parallelograms, on the ſame Baſe AD.

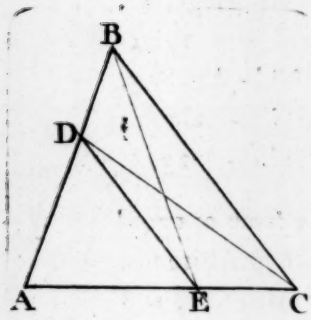
Draw the Perpendicular BH;-(Prob. 7.)
BH and GH are their Altitudes. - Def. 46.

Then the Parallelogram AC:AF::BH:GH.

For, ſince all Parallelograms having the ſame Baſe and Altitude are equal, the Parallelograms, AC, AF, are equal to Rectangles, whoſe Baſes are equal to AD, and their other Sides equal to BH and GH, reſpectively. - 18. 1.
Wherefore, BH and GH being conſidered as the Baſes of the Parallelograms AC and AF, and their Altitudes, AD;
the Parallelogram AC:AF::BH:GH - - Theo.
But, the Triangle ABD is half the Par. ABCD, } - 17. 1.
and, the Triangle AGD is half the Par. AEFD, }
Conſ. the Triangle ABD:AGD::BH:GH.-Ax. 13. 5.

THEOREM II. 2 Euclid.

If a Right Line be drawn, parallel to any Side of a Triangle; it will cut the other two Sides (produced if neceſſary) proportionally.



Firſt. In the Triangle ABC, let DE be drawn, parallel to BC; cuting the Sides AB and AC, in D and E. - (Prob. 5.)

Then will AB and AC, be cut proportionally, in D and E.

i. e. AD to DB, as AE is to EC.

Draw BE and DC.

Dem. Becauſe DE is parallel to BC, DBE = EDC - 18. 1.
Wh. the Triangle ADE:DBE::ADE:EDC. - Ax. 4 5.
But, the Triangle ADE:DBE:: AD : DB;
and, the Triangle ADE:EDC:: AE : EC. } - Th. 1.
Therefore, - - AD : DB :: AE : EC. - Ax. 13. 5.

Caſe 2. If ADE be the given Triangle; let BC be drawn (without it) parallel to DE.

Then, if the Sides AD, AE be produced, cuting BC, in B and C; the Sides, AD and AE, of the Triangle, ADE, are proportional to the Segments, DB and EC, of thoſe Sides, produced to B and C.

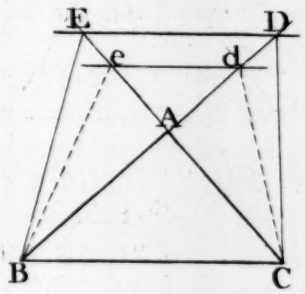
For, as AD : DB :: AE : EC; - - - as above.
conſ. as AD : AE :: DB : EC. - - 4. 5.

Cor. 1. If two Sides of a Triangle (AB, AC) are cut proportionally from the ſame Angle, (A, in D and E; ſo, that, AD:DB::AE:EC) a Right Line (DE) joining the Points (D and E) will be parallel to the remaining Side of the Triangle. The Converſe.

For,

For, the Sides, AB, AC, being cut, as AD:DB::AE:EC; and the Right Lines, BE, DC, drawn, as before;
$\triangle$. ADE:DBE::ADE:EDC; (1.) conſ. DBE=EDC.
Therefore, DE is parallel to BC; - - Cor. to 18. 1.

Caſe 3. If DE be drawn, beyond the Vertex A, parallel to BC, the Side oppoſite; and, if the Sides BA and CA be produced, till they cut DE, in D and E; then is AB to AD as AC to AE.



Join BE and DC; as before.
The Triangle ACD is equal to ABE.

Dem. DE is parallel to BC; wh. $\triangle$ DBE=DCE - 18. 1.
and, ADE is common to both; th. ACD=ABE - Ax. 7. 1.
Conſ. the Triangle ABC:ACD::ABC:ABE - Ax. 4. 5.
But, - - - ABC:ACD:: AB : AD;
and, - - - ABC:ABE :: AC : AE. } Theo. 1ſt.
Therefore, - - AC: AE :: AB : AD. - Ax. 13. 5.

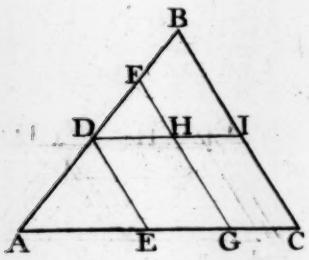
Cor. 2. Two Right Lines (BD and CE) cuting each other (in A) between, or beyond, two parallel Lines (DE and BC) are cut proportionally, in that Point, and the Points in which they cut the Parallels.

For, AB:AD::AC:AE, in the 3rd Caſe, directly; and alſo, in the other two, by compoſition. - Th. 6. 5.

Cor. 3. A Right Line drawn parallel to the Side of a Triangle (in the firſt Caſe) cuts off a Triangle, ADE, ſimilar to the whole Triangle ABC; and, in the other two Caſes, the Triangles ADE and ABC are alſo ſimilar. (Def. 1.)

For, the Angle ADE=ABC, and AED=ACB - 4. 1.
and, the Angle A is either common, or vertical.

COR. 4. If ſeveral Lines be drawn, within a Triangle, parallel to any Side; the correſponding Segments of the other two Sides, cut by the parallel Lines, are proportional; each to the other, reſpectively.



Let DE and FG be parallel to BC, in the Triangle ABC.

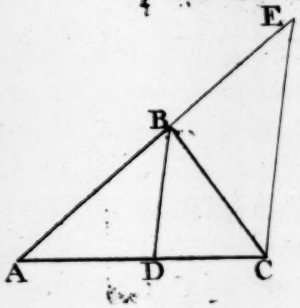
Then, AD : DF : FB :: AE : EG : GC.

For, the Segment AD : DF :: AE : EG. - Theorem.
Let DI be drawn parallel to AC; then, DF : FB :: DH : HI
But, EDHG and GHIC are Parallelograms, - by Conſtr.
wherefore, DH = EG, and HI = GC - - 15. 1.
Therefore, DF : FB :: EG : GC (as DH : HI) - Ax. 4. 5.
But, AD : DF :: AE : EG; th. AD : FB :: AE. GC - 9. 5.

THEOREM III. 3 Euclid.

If any Angle of a Triangle is biſected by a Right Line, cuting the oppoſite Side; the Segments of that Side, will be proportional to the two Sides of the Triangle, containing the Angle biſected.

And, converſely, if any Side of a Triangle be cut, in the proportion of the other two Sides; and if the greater Segment be contiguous to the greater Side; then will a Right Line, drawn from the point of ſection to the oppoſite Angle, biſect that Angle.



Let the Angle ABC, in the Triangle ABC be biſected by the Right Line BD, cutting the Side AC in D.

I ſay, AD is to DC, as AB is to BC.

DEM.

DEM. Produce AB indefinite; make BE=BC, and draw CE.
Then, becauſe BE=BC, the Angle BEC=BCE, - 9.1.
(for, the Triangle CBE is Iſoſceles)
and the Angle ABC (which is external)=BEC+BCE-10.1.
Now, the Angle ABD = DBC, by Conſtruction;
and BCE=BEC; conſequently, DBC=BCE. - Ax. 3.1.
But, the Angles DBC, BCE, are alternate; and ſince they are equal; conſequently, BD is parallel to CE - 4.1.
Wherefore, in the Triangle AEC, the Sides AC, AE, are cut proportionally, by the Right Line BD;
i. e. AD : DC :: AB : BE - - - - - Th. 2.
But, BE = BC (Con.) th. AD : DC :: AB : BC. - Ax. 4.5.

2nd. In the Triangle ABC, let the Side AC be ſo cut, in D, that the Segments, AD, DC, are in the Ratio of AB to BC.
Then, a Right Line, BD, biſects the Angle ABC
(the ſame Conſtruction remaining as before).

Now, BE=BC (Con.) and, AD:DC::AB:BC - Hyp.
wh. AD:DC::AB:BE; conſ. BD is parallel to CE - 2.
and conſequently, the Angle DBC= BCE - 4.1.
But, becauſe (in the Triangle CBE) BC = BE, - Con.
the Angle BCE = BEC; i. e. AEC. - - 9.1.
and, the Angle ABC = BCE + BEC. - - 10.1.
But, the Angle DBC = BCE; and ABD = AEC. - 4.1.
Therefore, ABD is equal to DBC. - - Ax. 3.1.

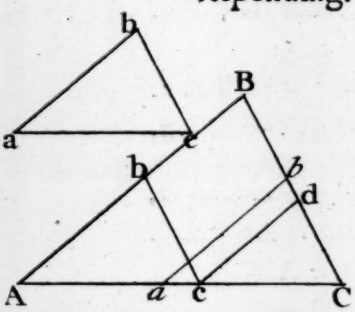
N. B. This Theorem and the foregoing are of extenſive utility, and are particularly uſeful in Perſpective.

The third Caſe of Theo. 2, and the following Corollary, contain the whole Subſtance of practical rectilinear Perſpective.

THE-

THEOREM IV. 4 Euclid.

In equiangular Triangles, the Sides containing equal Angles are proportional; and, the Sides which ſubtend equal Angles are homologous, or correſponding.



Let ABC, and a b c, be equiangular Triangles, having the Angle a, equal to the Angle A, b equal B, and c equal C.

I ſay, the Side a b, is in proportion to A B, as a c is to A C, and, as b c to BC.

DEM. Suppoſe the Angle a applied to, or laid upon the Angle A, and a b upon A B. - - - Poſt. 5. 1.
Then, becauſe the Angle b a c = B A C, a c will fall on AC, and b c will cut the Triangle ABC, in b c.
But, the Angle ABC = A b c; and are internal & oppoſite; wh. bc is parallel to BC (4.1.) conſ. Ab:bB::Ac:cC-2.
Wh. Ab:Ab+bB (equal AB)::Ac:Ac+cC (eq.AC)-6.5.
Therefore, as a b : AB :: ac : AC.

Again; ſuppoſe the Triangle a b c applied to the Angle C, of the Triangle ABC; (as *a b* C) then will the Side a b (i. e. *a b*) be parallel to AB. - - - Cor. 1. Th. 2.
The reſt is as before; i. e. *b* C : BC :: *a* C : AC.
But *a* C = A c, &c. and, Ac : AC :: Ab : AB, - above.
conſequently, *b* C (i. e. bc) : BC :: Ab (i. e. ab) : AB.

It is thus demonſtrated, according to Euclid.

Let

Let A b c and c d C be equiangular Triangles, ſo applied, at the Angle c, that the two Sides, A c and c C, are in a Right Line.

Produce A b and C d, meeting in B.

Then, becauſe the Angle A c b = A C d, & C c d = C A b, A B is parallel to c d; and, C B parallel to c b - 4. 1.
wherefore, c b B d is a Parallelogram; - Def. 33.
conſequently, b B = c d, and B d = b c. - - - 15. 1.
But, A b : b B :: A c : c C; - - Th. 2.
wherefore, A b : c d (equal b B) :: A c : c C; } Ax. 4 5.
alſo, as A c . c C :: b c (equal B d) : d C }
conſequently, as A b : c d :: b c : d C. - - - 13. 5.

Cor. 1. Triangles, having all their Sides directly proportional, are equiangular. The Converſe; needs no proof.

Cor. 2. Triangles, being equiangular, or having all their Sides directly proportional, are ſimilar. - ſee Def. 17.

Conſequently, if two Angles, of one Triangle, are equal, reſpectively, to two Angles of another Triangle, the Triangles are ſimilar; and their Sides proportional.

For, the two remaining Angles are equal - 10. 1.

Hence is deduced the 15th Problem, and alſo the 16th.

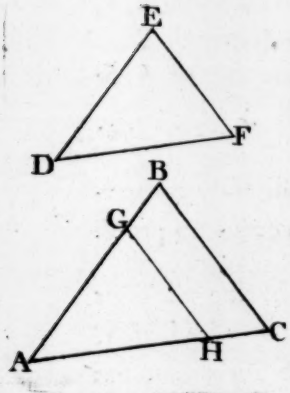
N. B. This Theorem is of univerſal application. It is the Criterion of Proportion in every branch of the Mathematics; for, wherever there is found equiangular Triangles, ſeeing that they are neceſſarily ſimilar, the Proportion of their correſponding Sides, is conſequently analogous.

The firſt Corollary, the Converſe of the Theorem, is the fifth Propoſition of Euclid.

THE-

THEOREM V. 6 Euclid.

If, in two Triangles, an Angle of one be equal to an Angle of the other, and, if the Sides containing the equal Angles are directly proportional, the Triangles are ſimilar.



In the Triangles ABC, DEF, let the Angle BAC be equal EDF; and, as AB is to DE, ſo let AC be to DF.

Let the Triangle DEF be applied to the Triangle ABC; the Angle D to the equal Angle A. - - Poſt. 5. 1.

Or, from the Angle A (equal D) take AG, equal DE, and AH equal DF, and join GH. - - - (Prob. 3.)

DEM. Now, becauſe AB : DE :: AC : DF; - (Hyp.)
conſequently, AB : AG :: AC : AH. - Ax. 5. 5.
wh. AB—AG (GB) : AG :: AC—AH (HC) : AH, - 7. 5.
i. e. GB : AG :: HC : AH; or, as AG : GB :: AH : HC - 5. 5.
wherefore, GH is parallel to BC; - - Cor. 1. 2. 6.
therefore, the Triangle AGH is ſimilar to ABC. - C. 3.
But, the Triangles AGH, DEF are congruous - 8. 1.
Therefore, the Triangle DEF is ſimilar to ABC-Axiom.

The 7th Propoſition of Euclid, is of little or no conſequence, as it ſeldom, if ever, occurs; it is rather a critical remark, than a neceſſary Propoſition.

THE-

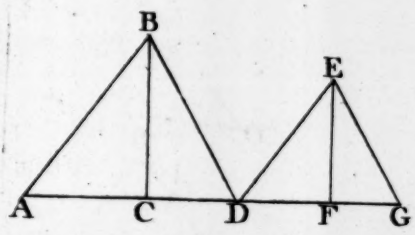
THEOREM VI.

In ſimilar Triangles, the Perpendiculars, from equal Angles, are proportional to the correſponding Sides.

And, correſponding Sides are cut proportionally by the Perpendiculars.

Let, the Triangles ABD, DEG be ſimilar, whoſe correſponding Sides are AB and DE, AD and DG, BD and EG.

I ſay, the Perpendiculars BC, EF, from the equal Angles, B and E, are proportional to the Sides.



Dem. For, the Triangles ABC, DEF, alſo CBD, FEG are ſimilar; becauſe, the Angles, at C and F, are all Right (Def. 11.) therefore equal; - - Ax. 9. 1.
the Angle A = EDF, and BDC = G; by Hypotheſis;
conſequently, ABC = DEF, and CBD = FEG. - 10. 1.

Wherefore, as AB : DE :: BC : EF.
But, - - AB : DE :: AD : DG. } - Th. 4.

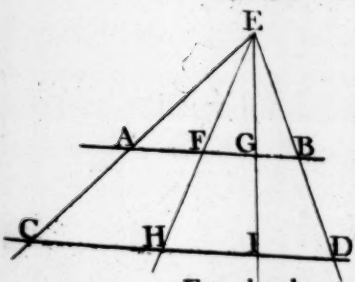
Therefore, - BC : EF :: AD : DG.
Alſo, - - BC : EF :: BD : EG. } - Ax. 13. 5.

2nd. AD and DG are cut proportionally, in C and F, by the Perpendiculars BC and EF.

Becauſe AC : BC :: DF : EF;
and, - - BC : CD :: EF : FG } - - Th. 4.
conſ. - as AC : CD :: DF : FG. - - - - 9. 5.

Cor. If parallel Lines are cut by any number of Lines, proceeding from the ſame Point, they will be cut proportionally.

And, the Lines which proceed from a Point will alſo be cut proportionally, by the parallel Lines.

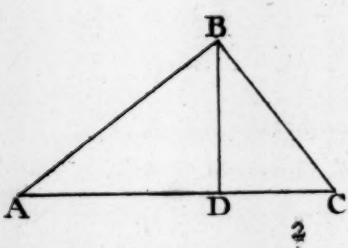


Let AB and CD be parallel Lines. From any Point, E, at pleaſure, if the Right Lines EA, EF, &c. are drawn, cuting the Parallels, in A and C, F and H &c; they will be cut proportionally, in thoſe Points.

For, in the △s. AEF, CEH, the Angle EAF=ECH-4.1.
alſo, AEF is common to both Triangles;
conſequently, the Triangles AEF and CEH are ſimilar.
And, for the ſame reaſon, the Triangles FEG and HEI;
alſo, GEB and IED are, reſpectively, ſimilar.
Wherefore, EA : EC :: EF : EH :: EG : EI :: EB : ED.
conſequently, AF : CH :: FG : HI, and as GB : ID.
Therefore, CH : HI : ID :: AF : FG : GB.
Alſo, EA : AC :: EF : FH :: EG : GI, and as EB : BD.
Hence is deduced the 36th, a moſt uſeful Problem.

THEOREM VII. 8 Euclid.

In a right angled Triangle, if a Perpendicular be drawn, from the Right Angle to the Hypothenuſe, it will divide that Triangle into two Triangles, which are ſimilar to each other; and alſo to the whole Triangle.



In the Right angled Triangle, ABC, and, from the Right angle, B, draw the Perpendicular BD, to AC.

I ſay, the Triangles ABD, DBC, are ſimilar, to each other, and alſo to the whole Triangle, ABC.

DEM.

DEM. The Angle ADB = ABC (Con.) and BAD is common; conſequently, the Angle ABD = BCD - - 10. 1.
Wherefore, the Triangles ABD, ABC are ſimilar - 4.
Again; the Angle ADB = BDC (Def. 10.1)
and ABD = BCD (above) wherefore, BAD = DBC - 10.1.
Conſequently, the Triangle ABD is ſimilar to DBC - 4.
But, the Triangle ABD is ſimilar to ABC - proved
Therefore, the Triangle DBC is ſimilar to ABC - Axiom.

COR. 1. In right angled Triangles, the Perpendicular (BD) is a mean Proportional between the Segments (AD & DC) of the Hypothenuſe, made by the Perpendicular.

For, the Triangles ABD, DBC, being ſimilar, the Side AD (of the Triangle ABD) is to BD (of the ſame) as the Side BD (of the Triangle D B C) is to D C (of the ſame)
i. e. AD : BD : : BD : DC. - - - - Th. 4.

From hence, and from Theorem 12 of the 3rd Book, are deduced the 30 and 31ſt Problems, in Practical Geometry; which are of moſt extenſive utility.

COR. 2. In right angled Triangles, each Side containing the Right Angle, is a mean Proportional, between the adjoining Segment, and the whole Hypothenuſe.

AB is a mean Proportioal between A D and AC;
and B C is a Mean, between D C and A C.

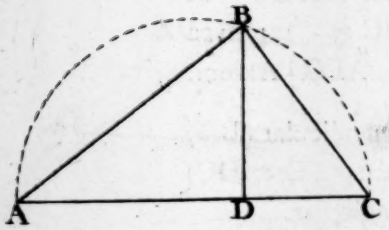
For, AD : AB (of the Tri. ABD) : : AB : AC, of the Tri. ABC
And, AC : BC (of the Tri. ABC) : : BC : DC, of the Tri. DBC } - 4.

COR. 3. A Perpendicular, from any Point in the circumference of a Circle, to a Diameter, is a mean Proportional, between the two Segments of the Diameter, made by the Perpendicular.

For every Angle, touching the Circumference, in a Semicircle, is a Right one (12. 3.) conſequently, the Diameter is the Hypothenuſe of a right angled Triangle, inſcribed in the Circle.

From hence, may eaſily be deduced the following Problems:

PROBLEM I. *The two Sides, or Legs, of a right angled Triangle being given, how to find the Hypothenuſe, the Perpendicular, and the two Segments of the Hypothenuſe, arithmetically.*



In the right angled Triangle, A B C, is given the two Legs, A B and B C.

1ſt. To find the Hypothenuſe.

A B □ + B C □, = A C □ - 20. 1.

Conſequently, the Squares of A B and B C, being added together, the Square Root, of that Product, gives A C.

2nd. To find the Perpendicular, BD; AC: AB:: BC: BD - 4.

BD is, therefore, a fourth Proportional; AC, AB, and BC; being the three Terms given, and BD is required - Def. 11. 5. Conſequently, AB × BC = AC × BD - - (See Th. 9.) i. e. if A B be multiplied by B C, and that Product divided by A C (as in the Rule of Three) the Quotient ariſing is B D.

For, the Diviſor multiplied into the Quotient is equal to the Dividend; i. e. A C multiplied into B D. is equal to AB multiplied into B C; by Theorem 9th.

Therefore, the Perpendicular, B D, is a fourth Proportional.

Hence, a Rectangle, under the two Legs of a right angled Triangle, is equal to a Rectangle under the Hypothenuſe and the Perpendicular.

3rd. To find the greater Segment, AD; AC: AB:: AB: AD-4.

Wherefore, AD, the greater Segment of the Hypothenuſe, is a third proportional to A C and A B. - - See Def. 9. 5. Conſ. AB □ = AC × AD; or, the Rect. CAD (ſee Cor. Th. 9.) i. e. A B multiplied by itſelf, and the Product divided by A C, gives A D.

Otherwiſe; BC : BD :: AB : AD.
Conſ. AD is a fourth Ptoportional; and AB × BD ÷ BC = AD.

4th. To find the leſſer Segment, DC; AC: BC :: BC: DC; or AB : BC :: BD : DC. Alſo, AD : BD :: BD : DC.

When either Segment, AD or DC, is found, the other is a third Proportional, between that Segment and the Perpendicular, BD; conſ. AD × DC = BD □. But, if the Hypothenuſe, A C, and either Segment be found, then, AC − AD is equal D C; and AC − DC = A D.

PROB. II. *The Perpendicular and either Side being given, how to find the other Side.*

Let A B and B D be given.

Becauſe A B □ = AD □ + B D □ ; - - 20. 1.
conſequently, A B □ — B D □ = A D □.
i. e. the ſquare of BD being ſubtracted from the ſquare of AB, the ſquare Root of the remainder gives AD.
Then, as AD : AB : : BD : BC ; wh. BC is a fourth Proportional.
Conſ. AB × BD ÷ AD = BC. For A B × BD = AD × B C.

PROB. III. *The Perpendicular and either Segment being given, to find the other Segment, and the two Sides.*

Let AD be the given Segment.

Then, as AD : BD : : BD : DC. Wh. DC is a third Proportional.
Conſ. B D multiplied into itſelf, and that Product divided by AD, gives DC. For AD × DC = BD □.
If DC had been given, it is juſt the reverſe.

2nd. AD □ + DB □ = AB □. And BD □ + DC □ = BC □ - 20. 1.
Wherefore, if the ſquare of AD be added to the ſquare of BD, the ſquare Root of that Sum gives the Side AB.

Again ; AB being found, BC is a fourth Proportional.
For, AD : BD : : AB : BC. Wherefore, AB × BD ÷ AD = BC.
i.e. the ſum of AB multiplied by BD, divided by AD gives BC.

PROB. IV. *The two Segments* (AD *and* DC) *being given, to find the Perpendicular* (BD) &c.

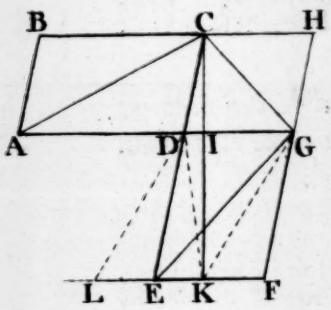
BD is a mean Proportional between the Segments AD & DC.
Wherefore, as AD : BD . : BD : DC ;
conſequently, AD × DC = BD □. - - (ſee Cor. Th. 9.)
i. e. If AD be multiplied by DC, the ſquare Root of that Product, is BD.
Hence, the Sides, AB and BC, may be found, as in Problem 3.

Theſe Problems, deduced from this extraordinary and extenſive Theorem, are extremely uſeful in Perſpective ; to find Vaniſhing Points and their Diſtances, &c.

THEO-

THEOREM VIII. 14 & 15 Euclid, reverſed.

Parallelograms, having equal Angles; or, Triangles, having one Angle, in each, equal to one another; and the Sides, containing the equal Angles, reciprocally proportional, are equal.



Let ABCD and DEFG be Parallelograms, having equal Angles; and as the Side AD, of the one, is to DG, of the other, ſo let DE, of the ſame, be to DC of the firſt.

Let the equal Angles, at D, be ſo placed together, that, the Side AD, of one Parallelogram, and DG of the other, are, in one Right Line; then will DE and DC be alſo in one Right Line. - 2. 1.

Produce BC and FG, meeting in H, forming a Parallelogram CDGH.

DEM. Then, the Par. BD is to DH, as AD is to DG } T. 1.
and, the Parallelm. DF is to DH, as ED is to DC }
But, AD : DG :: ED : DC; by Hypotheſis;
wherefore, the Parallelograms, BD and DF, have an equal Ratio to the ſame Parallelogram, DH.
Therefore, the Parallelogram ABCD = DEFG - Ax. 5. 5.

2nd. Having drawn the Diagonals AC and EG; the Triangle ACD is equal to DEG; - - - Ax. 4. 1.
for, each is equal to half the Parallelogram BD or DF - 17. 1.
Or (CG being drawn) the Tri. ACD : DCG :: AD : DG,
and, - - - the Triangle DEG : DCG :: ED : DC.
But, AD : DG :: ED : DC (Hyp.) conſ. Tri. ACD = DEG,
for, they have an equal Ratio to the Triangle DCG.

Cor. 1.

Cor. 1. Equal Parallelograms having equal Angles, or Triangles having one Angle in each alſo equal, have their Sides about the equal Angles reciprocally proportional.

This, being the converſe of the Theorem is manifeſt, from it; for, by the ſame conſtruction, it is thus reverſed.

Becauſe, the Parallelogram BD is equal to DF, they have an equal Ratio to the Parallelogram DH - - Ax. 4. 5.
But, Par. BD:DH::AD:DG; and, Par. DF:DH::ED:DC - 1.
Therefore, as AD : DG :: ED : DC - - - - - Ax. 13. 5.
Alſo, the Triangle ACD is equal to half the Par. BD; and DEG is equal to half DF (17. 1.) conſequently, ACD=DEG.-Ax. 4. 1. and, their Sides, AD, DC; DE, DG, are reciprocally proportional.

Cor. 2. Parallelograms, or Triangles, having their Baſes and Altitudes reciprocally proportional, are equal.

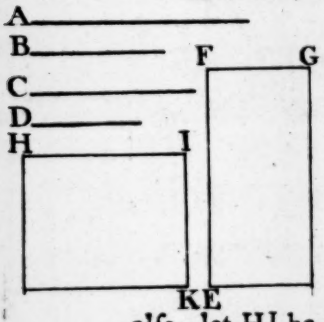
For, whether their Angles be equal or not, the Parallelograms are equal to Rectangles having the ſame Baſe and Altitude - 18. 1.
Conſequently, if their Baſes (AD and EF) and their Altitudes (CI and IK) are reciprocally proportional, the Parallelograms, BD and DF, or the Triangles, ACD, DEG, are equal - Th. 1.

For, the Perpendiculars (i. e. the Altitudes) of Triangles and Parallelograms, having equal Angles, are in the ſame Ratio as their Sides; i. e. CI : IK :: CD : DE - - - Th. 6, or 2.
But, their Baſes, EF and AD, are in the ſame Ratio.
Therefore, CI : IK :: EF, or KL (equal DG) : AD; - Ax. 13. 5.
and conſequently, whether the Angles, at D, are equal or not,
the Parallelogram ABCD=DGKL (equal DEFG.)]
Alſo, the Triangle DKG=ADC (equal DEG.)] - 18. 1.

THEO-

THEOREM IX. 16 Euclid.

If four Right Lines are Proportionals, the Rectangle under the two Extremes (i. e. the greateſt and the leaſt) is equal to the Rectangle contained under the two Means.

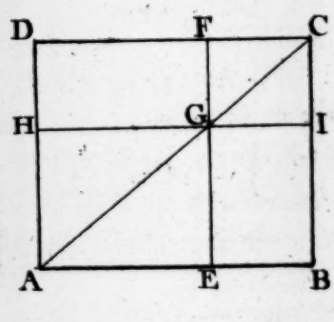


Let the four Right Lines, A, B, C, D, be proportional; and as A is to B, ſo let C be to D.

Then will the Rectangle under A and D, be equal to that under B and C.

Conſtruct the Rectangles EG and HK, making EF equal A, and FG equal D; alſo, let HI be equal C, and IK equal B. (Prob. 18.)

Dem. Then, becauſe EG and HK are Rectangles, conſequently their Angles are all equal. (Def. 34.)
But, as EF : IK :: IH : FG; i. e. as A : B :: C : D; wherefore, their Sides are reciprocally proportional.
Therefore, the Rect. EG, is equal to the Rect. HK. - 8.



Otherwiſe:

Take AB equal to the greateſt of the four proportional Lines, and, at either Extreme, B, make a Right Angle, ABC.

Make BC equal to the greateſt of the Means; and AE equal to the other.

Compleat the Rectangle ABCD, and draw the Diagonal, AC. - - (Prob. 18.)
Thro' E draw EF, parallel to BC; and through G, where it cuts the Diagonal, draw HI parallel to AB. (Prob. 5.)

Dem.

Dem. Now, becauſe AB, BC, and AE are equal to three of the Proportionals, EG is equal to the fourth;
for, as AB is to BC ſo is AE to EG. - - Th. 4.
And, ſince the Complements, DG, GB, are equal - 19. 1.
if HE be added to both, the Rect. ADFE = AHIB - Ax. 6. 1.
But, AHIB is a Rect. under the two Extremes.
And, ADFE is a Rect. under the two Means. } - Con.
Therefore, the Rectangle under the two Extremes, of four proportional Lines, is equal to a Rectangle under the Means.

Corollary. The 17th Propoſition of Euclid.

If three Right Lines are Proportionals, the Rectangle under the two Extremes, is equal to the Square of the Mean.

If the two middle Terms, B and C, had been equal, the Rectangle HK would be a Square. (Def. 35.) See Fig. 1.
But, as EF : IK :: IH : FG; and IH is equal to IK. - Sup.
Therefore, the Rectangle under the Extremes, of three proportional Lines, is equal to the Square of the Mean.

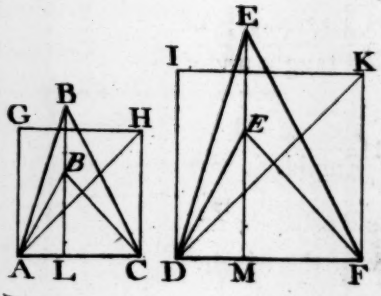
Or, if (by the 2nd Conſtruction) AE had been equal to BC, the Rectangle ADFE would be a Square.
Conſ. it would be equal to the Rect. AHIB under the two Extremes. For, as AB : AE (eq. BC Sup.) :: AE : EG, eq. BI.

From hence, and Prop. 19. 1. is deduced the laſt method (Prob. 32) for finding a fourth Proportional.
A third Proportional may alſo be found, by the ſame; viz. by conſtructing a Square on either of the given Lines, which is to be the Mean.
From which Conſtructions, it is obvious, is deduced that moſt excellent, Golden Rule, or Rule of Proportion, in Arithmetic.

N. B. If a Right Line be ſo divided in two Parts, that the Rectangle, under the whole Line and one Segment, is equal to the Square of the other; it is divided in extreme and mean Proportion in that Point.
For, the whole Line, the greater Segment, and the leſs, are in continual Ratio; conſequently, they are three Proportionals.

THEOREM X.

Similar Triangles are proportional to the Squares of their correſponding Sides.



Let ABC, DEF be ſimilar Triangles, having the Angles at A and D equal; the Angle B equal E, and C equal F.

Then, Squares conſtructed, on correſponding Sides, AC and DF, or any other, have the ſame Ratio to each other, reſpectively, as the Triangles, ABC and DEF.

Deſcribe the Squares AGHC and DIKF, on the Sides AC and DF, which are homologous; being oppoſite equal Angles, B and E.

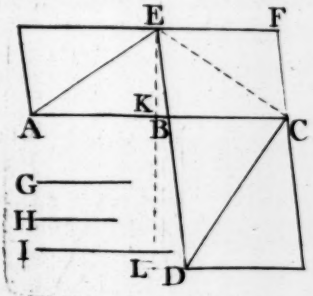
Draw the Diagonals AH, and DK; alſo the Perpendiculars BL, and EM, to the Sides AC, and DF.

DEM. The △ABC:AHC::BL:HC (eq. AC, Con) } C. Th. 1.
Alſo, the △DEF:DKF::EM:KF, equal DF. }
But, - as BL:AC::EM:DF; - - Th. 6.
wherefore, △ABC:AHC::DEF:DKF - Ax. 13. 5.
conſ. - △ABC:DEF::AHC:DKF - - Th. 4. 5.
But, △AHC:□AGHC::△DKF:□DIKF-17. 1. & Ax 7. 5.
Wh. △ABC:□AGHC::△DEF:□DIKF - Poſt. 1.
Th. △ABC:DEF::□AGHC:DIKF - Th. 4. 5.
That is, the △ABC:DEF::AC□:DF□. Q. E. D.

THE-

THEOREM XI. 23 Euclid.

Equiangular Parallelograms are, to one another, in a Ratio which is compounded of the Ratio of their Sides.



Let AE and DC be equiangular Parallelograms.

I ſay, the Ratio of the Parallelogram AE to DC, is equal to the compounded Ratio of their Sides. i. e. of the Ratio of AB to BC, and of EB to BD.

Let the equal Angles, at B, be ſo placed together, that, the Sides, AB, and BC, alſo DB and BE, are in a Right Line; and produce the oppoſite Sides, meeting at F, forming a Parallelogram BEFC.

Take any Right Line, G, at pleaſure; and, as BC is to AB, make H to G;
alſo, as DB is to BE, make I to H. (Prob. 32.)

Dem. Now, the Ratio of G to H is the ſame as AB to BC; and, the Ratio of H to I is equal to that of EB to BD.-Con. But, the Ratio of G to I, is compounded of the Ratios of G to H, and H to I (Def. 21. 5.) conſequently, the Ratio of G to I is that, compounded of the Ratios of the Sides; to which, theſe are reſpectively equal.

But, the Par. AE:EC::AB:BC; i.e. as G to H.
And, the Par. EC:DC::EB:BD; i.e. as H to I. } - Th. 1.

Wh ſince the Par. AE:EC::G:H; and Par. EC:DC::H:I.
The Parallelogram, AE: Par. DC::G:I. - Th. 9. 5.
But, G is to I, in the compounded Ratio of the Sides.
Therefore, the Par. AE is to Par. DC in the Ratio, which is compounded of the Ratio of their Sides. Q. E. D.

COR. 1. Triangles, having one Angle in each, equal to one another, have that Ratio between them, which is compounded of the Ratio of the Sides, containing equal Angles.

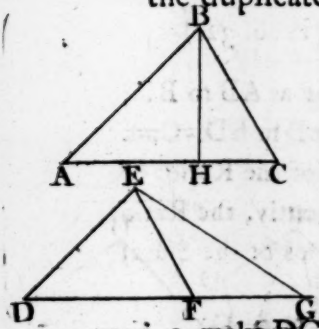
For, the Triangles AEB, BCD, having equal Angles, at B, and drawing EC, the Demonſtration is in the Theorem.

COR. 2. Parallelograms, having equal Angles, or Triangles, which have one Angle in each, equal to to one another, have that proportion, to each other, as the Rectangles under the Sides containing the equal Angles.

For, the Sides, containing equal Angles, have the ſame proportion, to each other, as the Perpendiculars, EK, KL.- Th. 6. Conſequently, Rectangles under the Baſes and Perpendiculars, are in the ſame Ratio to each other, as Rectangles under the Baſes and adjoining Sides, containing equal Angles, viz. in the Ratio which is compounded of their Sides; by Theorem.

THEOREM XII. 19 Euclid.

The proportion of ſimilar Triangles, to each other, is the duplicate Ratio of their correſponding Sides.



Let the Triangles ABC, DEF be ſimilar and alike ſituated; i. e. let the Angle A be equal D, B equal E, and C equal F; The Side AB correſponds with DE, BC with EF, and AC with DF. Produce DF. Take DG a third Proportional, to DF and AC, and draw EG; i. e. make DG to AC as AC is to DF; (Prob. 31.) Then, the Triangle ABC is to DEF, as DG to DF.

DEM. Becauſe the Triangles ABC, DEF are ſimilar, conſequently, AB : DE :: AC : DF. - - Th. 4. But, DG : AC :: AC : DF (Con) conſ. as AB : DE :: DG : AC. Wherefore, in the Triangles ABC, DEG, the Sides AB and AC, DE and DG, which are about the equal Angles (A and D) are reciprocally proportional. conſequently, the Triangle DEG = ABC. - Th. 8.

But, the Triangle DEF:DEG::DF:DG - - Th. 1.
Th. the Triangle DEF:ABC::DF:DG - Ax. 4. 5.
and, by inverſion, ABC:DEF::DG:DF. - Th. 5. 5.
i. e. in a duplicate Ratio of their correſponding Sides.

Or, if AH be taken a third proportional to AC and DF; it will then be, as AB:DE::DF:AH, i.e.::AC:DF-Con.
Conſequently, the Triangle ABH = DEF, - Th. 8.
Wherefore, the Tri. DEF:ABC::ABH:ABC.- Ax 4.5.
i. e. as AH is to AC (Th. 1.) viz. in a duplicate Ratio.

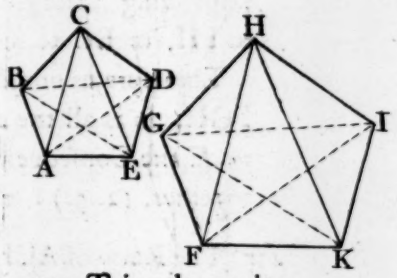
Cor. If three Right Lines are Proportionals, the Squares of the firſt to the ſecond are as the firſt Line is to the third.

In the Tri. DEF, ABC, and DEG, DF:AC:DG ÷ - Con.
But, the Triangle DEF:ABC::DF:DG. - - Theo.
And the Triangle DEF:ABC::DF□:AC□. - - - 10.
Therefore, as DF:DG (the firſt to the third) ::DF□:AC□.

THEOREM XIII. 20 Euclid.

Similar Quadrilaterals, and Poligons, are, by drawing Diagonals, divided into Triangles, equal in Number; the correſponding Triangles are ſimilar to each other, reſpectively; and proportional to the Poligons.

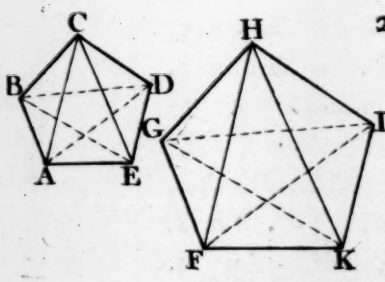
Alſo, the proportion of the Poligons, to each other, is in the duplicate Ratio of their correſponding Sides.



Let ABCDE and FGHIK be ſimilar Poligons.

Draw the Diagonals AC, CE, and FH, HK, &c. from any correſponding Angles, C and H, to the oppoſite.

Dem. Every Poligon is divided into as many Triangles, as it has Sides, wanting two; by drawing all the Diagonals which can be drawn, from any Angle; as CA, CE, or HF, HK; conſequently, the number of Triangles, in each, are equal; ſeeing the Poligons have an equal number of Sides.



2nd. Becauſe the Poligons are ſimilar, they are conſequently equiangular (Def. 1.) and, being ſimilarly poſited, the Angle A = F, B = G, and C = H &c.

Then, becauſe AB : BC :: FG : GH, and the Angle B = G, the Triangles ABC, FGH are ſimilar. - - Th. 5.

And, becauſe the Angle BAE = GFK and there is taken away, from each, equal Angles BAC, GFH, the remainder, CAE = HFK. - - Ax. 7. 1.

But, AB : AC :: FG : FH, and AB : AE :: FG : FK. conſequently, AC : AE :: FH : FK. - - 13. 5. therefore, the Triangles ACE, FHK are ſimilar. Th. 5.

And, becauſe CD : DE :: HI : IK, and the Angles D and I are equal; the Triangles ECD, KHI are ſimilar. - ſame.

3rd. Becauſe the Triangles ABC and FGH are ſimilar, their Ratio is dupplicate of their correſponding Sides; e. g. of AC to FH. - - - - Th. 12.

And for the ſame reaſon, the Ratio of ACE to FHK is alſo duplicate of AC to FH; or, of EC to KH. - ſame.

Alſo, the Ratio of ECD to KHI, is duplicate of EC to KH.

But, the Ratio of each Triangle, ABC, &c. to its correſponding Triangle, FGH, &c. is the ſame, viz. as AC to FH, or EC to KH, &c.

Therefore, as one Antecedent, ABC, is to one Conſequent, FGH, ſo is all the Antecedents ABC, ACE, and ECD, to all the Conſequents, FGH, FHK, and KHI, taken together, (2. 5.) i. e. as Poligon to Poligon. Q. E. D.

4th. The Ratio of ACE to FHK is duplicate of AE to FK. 12

But, the Ratio of Poligon to Poligon is equal ACE to FHK.

Therefore, the Ratio of Poligon to Poligon, is duplicate, of AE to FK, of ED to KI; or, of any other Sides, which are homologous.

4th.

Cor. 1. All ordinate or regular Figures whatever, Triangles, Squares, or Poligons of any kind are, reſpectively, to each other, in the duplicate Ratio of their Sides, or Diagonals.

Becauſe, all regular Figures, of the ſame Species, are ſimilar. Alſo, Circles are, to each other, in the duplicate Ratio of their Diameters, &c.

Hence, in all ſimilar Figures whatever, if the ratio of the homologous Sides be known, the proportion of the Figures is known.

For example; If the ratio of the Sides be as 2 to 3, their Areas are as 4 to 9. For, as $2 : 3 :: 3 : 4\frac{1}{2}$;
Therefore, as 2 is to $4\frac{1}{2}$, ſo is one Figure to the other;
which, in whole numbers, is as 4 to 9.

Cor. 2. The Proportion of all ſimilar Figures, whatever, are as the Squares of their correſponding Sides.

For, as the Triangle ABC or ACE, is to FGH, or FHK, ſo is Poligon to Poligon.
But, the Triangles, being ſimilar, are, as the Squares of their homologous Sides; viz. as AB to FG, or AE to FK - Th. 10.
Therefore, as the Square of AB is to FG, or of AE to FK, &c. ſo is Poligon to Poligon. - - - - Ax. 13. 5.

Cor. 3. The Perimeters or Circuits of ſimilar Figures are, to each other, as their correſponding Sides or Diagonals.

For, ſince each Side AB, BC, &c. has the ſame Ratio to its correſponding Side FG, GH, &c; alſo, as Side is to Side, ſo is Diagonal to Diagonal, AC to FH, &c; conſequently, as any one Side, or Diagonal, is to its correſponding Side or Diagonal, ſo is the ſum of all the Sides AB + BC + CD, &c. to the ſum of all the Sides, FG + GH + HI, &c. - 2. 5.

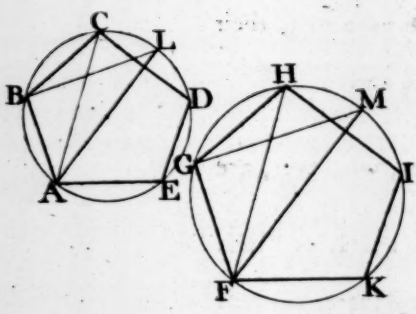
Cor. 4. Any ſimilar Figures whatever, deſcribed on the Mean and either Extreme of three proportional Lines, have the ſame Ratio to each other, as the two Extremes; i. e. as the firſt to the third.

For, they are to each other, in the duplicate Ratio of their correſponding Sides; by the Theorem.

From hence is deduced, an excellent Problem, (37) for finding the Side of any Figure whatever, ſimilar to another, and in any Ratio.

THEOREM XIV.

The Perimeters of ſimilar Poligons, inſcribed in Circles, are, to each other, in the ſame Ratio as the Diameters; and their Areas as the Squares of the Diameters.



Let ABCDE and FGHIK be ſimilar Poligons, inſcribed in Circles; whoſe Diameters are AL, and FM.

Then, as AL is to FM, ſo is the Perimeter ABCDE to FGHIK.

Alſo, as AL ſquare, is to FM ſquare, ſo is the Area of the Poligon ACE to FHK.

Join AC and BL, alſo FH and GM.

DEM. Then, the Angle BCA = BLA, and GHF = GMF - 10.3.
But, the Angle BCA = GHF (13) Th. BLA = GMF - Ax. 3.1.
and, the Angle ABL = FGM (for they are R. Angles) - 12.3.
therefore, the Triangles BLA, GMF are ſimilar; - Th. 4.
(for, the three Angles, are reſpectively equal)
Then, as AB : FG :: AL : FM. - - - - 4.
conſ. as AB □ : FG □ :: AL □ : FM □. - 10.
But, as AB : FG :: Perimeter ABCDE : Per. FGHIK - 2.5.
Th. the Diameter AL : FM :: ABCDE : FGHIK - Ax. 13.5.
Alſo, as AB □ : FG □ :: the Area of ACE : FHK - Th. 10 & 13.
and conſ. as AL □ : FM □ :: the Area of ACE : FHK - Ax. 13.5.

COR. The Circumferences of Circles are proportional to their Diameters, and their Areas to the Squares of their Diameters.

For, the Circumference of a Circle being conſidered as the Perimeter of a Poligon, of an infinite number of Sides, and, the Diameter may alſo be conſidered as a Diagonal; wherefore, the Demonſtration evidently follows from the Theorem.

COR. 2.

Cor. 2. Circumferences of Circles are proportional, to the Perimeters of ſimilar inſcribed, or circumſcribing, rectilinear Figures.

For the Perimeters, of all ſimilar right lined Figures, are in proportion to their correſponding Sides or Diagonals. - C. 3. 13. And, the Circumferences of Circles, are as their Diameters. - Th.

Cor. 3. The Areas of Circles are in proportion to that of any ſimilar inſcribed or circumſcribing Figures.

For, the areas of all ſimilar Figures, are as the duplicate ratio of their correſponding Sides or Diagonals; and, the areas of Circles are as the duplicate Ratio of their Diameters. - Th. 13, Wherefore, ſince the Diagonals of ſimilar Figures are in the ſame Ratio as their Sides, reſpectively; and alſo, as the Diameters of circumſcribing Circles; conſequently, the areas of Circles are, as the areas of ſimilar Poligons, inſcribed or circumſcribed.

Cor. 4. The area of a Circle is to that of any circumſcribed right lined Figure, as the Circumference of the Circle to the Perimeter of the circumſcribing Figure.

For, the area of a Circle is equal to a Triangle, whoſe Baſe is equal to the Circumference, and height, equal to the Radius. And, the Area of a circumſcribing Poligon, is equal to a Triangle, whoſe Baſe is equal to the Perimeter of the Poligon, and its height equal to the Radius of the inſcribed Circle. * Conſequently, the Areas of all circumſcribing Figures are as their Perimeters; which may be conſidered as the Baſes of Triangles of equal height. - - - - Th. 1. Therefore, the Area of a Circle is to that of a circumſcribed Poligon, as the Circumference of the Circle to the Perimeter of the Poligon.

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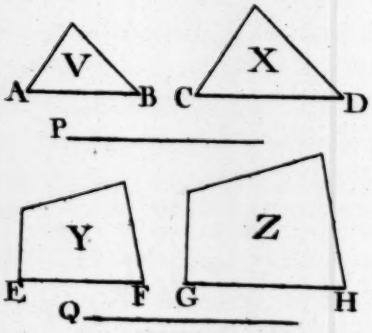
* See, Theory of Menſuration; Article 7, and 8.

THEOREM XV. 22 Euclid.

If four Right Lines are Proportionals; ſimilar Right lined Figures, deſcribed on each Pair, will alſo be proportional.



Let the Right Lines AB, CD, EF, and GH be proportionals, and let AB be to CD, as EF to GH.

On AB and CD, let there be conſtructed ſimilar Triangles, V & X; and, on EF and GH any other ſimilar Figures, whatever, Y & Z.

Find P and Q each a third Proportional to AB and CD, and, to EF and GH, reſpectively. (Prob. 31.)

DEM. Now, the Triangle V is to X, as AB to P; } -Th. 12.
and, the Trapezium Y is to Z, as EF to Q. }
But, as AB:CD::EF:GH; and, as CD:P::GH:Q -Con.
wherefore, as AB:P::EF:Q (9.5.) Th. V:X::Y:Z.

It is alſo true, if ſimilar Figures are conſtructed on the firſt and third, and others on the ſecond and fourth, of four proportional Lines.

For, ſince V:X:Y:Z, conſ. V:Y::X:Z. - Th. 4. 5.

COR. Similar Figures being proportional to other ſimilar Figures, their correſponding Sides are Proportionals.

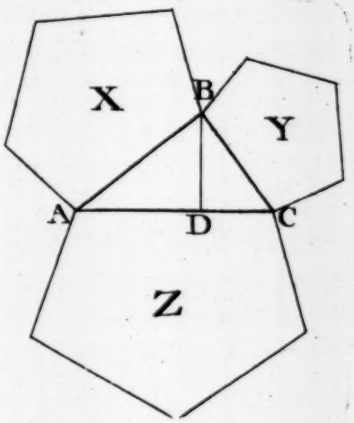
The Converſe, needs no proof; ſeeing it is but reverſing the Premiſes.

THE-

THEOREM XVI. 31 Euclid.

Any ſimilar Figures whatever, deſcribed on the three Sides of a Right angled Triangle, and if they are made correſponding Sides of the Figures; then, that which is deſcribed on the Side ſubtending the Right Angle, will be equal to both the others, deſcribed on the Sides containing the Right Angle.

Let ABC be a Right angled Triangle; and, let X, Y, and Z be ſimilar, irregular Pentagons, deſcribed on the three Sides, AB, BC, and AC, of the Triangle; which are correſponding Sides of the Figures, X, Y, and Z.



I ſay, that the Figure Z, deſcribed on the Hypothenuſe, AC, is equal to both X and Y, deſcribed on the other two Sides.

Draw the Perpendicular BD.

DEM. In every Right angled Triangle (A B C), ſince AD : AB :: AB : AC; and DC : BC :: BC : AC - Cor. to 7, Conſequently, any Figure, deſcribed on AB, is to a ſimilar Figure, deſcribed on AC, as AD is to AC.
and a ſimilar Figure, deſcribed on BC, is to that deſcribed on AC, as DC is to AC. - - - 12 and 13.
Therefore, ſince Z : X :: AC : AD; and Z : Y :: AC : DC, Z will be to X + Y :: AC : AD + DC. - Cor. to 3. 5.
But, AC = AD + DC; therefore, Z = X + Y. Q E.D.

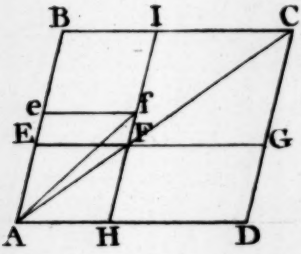
SHOL. *This Propoſition, by the Doctrine of Proportion, extends the famous Pythagorean Propoſition,* viz. *the* 47*th of the firſt Book of Euclid, and the* 20*th, of the firſt, of theſe Elements, univerſally.*

Seeing, by Th. 10. *of this* 6*th Book it is demonſtrated, that Triangles, and all ſimilar Figures whatever, are in proportion to the Squares of their correſponding Sides, the full and perfect demonſtration of this Propoſition neceſſarily follows from thence, and the* 20*th of the firſt; this Propoſition is therefore, in ſome meaſure, unneceſſary; but, for the particular beauty and elegance of it, I did not think proper to omit it.*

This Propoſition is alſo applicable to the Diagonals of Right lined Figures, as well as to their Sides; and, conſequently, to the Diameters of Circles. Wherefore, a Circle deſcribed on the Hypothenuſe of a Right angled Triangle, for its Diameter, is equal to the two Circles, whoſe Diameters are reſpectively equal to its Legs.

THEOREM XVII. 24 Euclid.

In every Parallelogram, thoſe which are about the ſame diagonal Diameter, are ſimilar to the whole Parallelogram and alſo to each other. And the two Complements have their Sides reciprocally proportional.



In the Parallelogram ABCD, let AEFH and FICG be Parallelograms, about the Diameter AC.

Then, the Parallelograms EH, IG, are ſimilar to the whole Parallelogram, BD, and to each other.

EG is parallel to AD, and BC; and HI, to AB and DC; by Conſtruction.

DEM.

Dem. Now, becauſe E F is parallel to BC, the Sides AB and AC, of the Tri. ABC, are cut proportionally.
wherefore, AE : EB :: AF : FC - - Th. 2.
and conſ. AE : AB :: AF : AC. - - - Conv. 6. 5.
And, becauſe HF is parallel to DC, as AF : AC :: AH : AD.
wh. as AE : AB :: AH : AD; and, as AE : AH :: AB : AD - 4. 5
But, EF = AH, and BC = AD, &c. (15. 1.) wherefore, the Sides of the Parallelograms A E F H, and A B C D, have all their Sides proportional.

But, the Angle EAH is common to both: and, the oppoſite Angles of Parallelograms are equal - - 15. 1.
wherefore, E F H = B C D. - - - Ax. 3.
Alſo, the Angle AEF = ABC, and AHF = ADC. - 4. 1.
therefore AEFH is ſimilar to the whole Par. ABCD - Def. 1.

By the ſame Reaſoning, the Parellelogram FICG may be proved ſimilar to A B C D.

For, CI : CB :: CG : CD; viz. as CF to CA - as above.
they are alſo equiangular; the Angle at C being common.
Conſequently, the Par. FICG is ſimilar to AEFH; being both ſimilar to the whole Parallelogram, ABCD - Axiom.

2nd. The Compliments BF, FD are equal. - - 19. 1.
and they are equiangular, (15. 1.) for EFI = HFG - 2. 1.
Wherefore, as EF : FG :: HF : FI, reciprocally - Th 8.

Corollary. 26 Propoſition of Euclid.

Hence it is evident, that, if on any Angle of a Parallelogram, there be deſcribed, or taken away, a leſſer Parallelogram, ſimilar, and a like ſituated, they will have the ſame common Diameter.

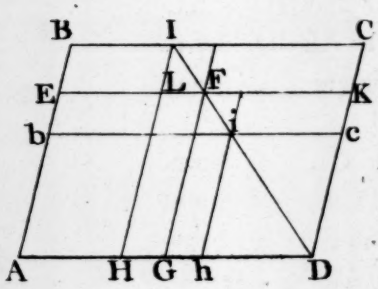
For, the Diameter AF or FC, is common with AC.

If any other Parallelogram A e f H, ſimilar to ABCD, be deſcribed at the Angle A, not alike ſituated, they have not the ſame Diameter AC.

THEO-

THEOREM XVIII. 27 Euclid.

If a Parallelogram be deſcribed on a Right Line, and from it there be taken away a Parallelogram, ſimilar, and alike ſituated, to one deſcribed on half the Line, equiangular to the firſt; that, which is deſcribed on the half Line, is greater than the remaining Parallelogram.



Let ABCD be a Parallelogram deſcribed on a given Right Line, AD; and, let AEFG be an equiangular Parallelogram, deſcribed on half the Line, AG; alſo, let HICD be taken away from the Parallelogram ABCD, ſimilar, and alike ſituated, to the Parallelogram AEFG.

I ſay, the Parallelogram AEFG is greater than the remaining Parallelogram ABIH.

Draw the Diameter ID, and produce EF to K.

DEM. Then, becauſe the Parallelograms, HC and GK, are about the ſame Diameter, ID, they are ſimilar. - 17. wherefore, AEFG is ſimilar to HICD. - - Axiom.

Now, HF = FC (19. 1.) and BF = FC - Th. 1. conſ. BF = HF (Ax. 3. 1.) and HF is greater than BL. add AL, to both; and AF is alſo greater than AI. - 8. Therefore, AEFG is greater than the remaining Parallelogram, ABIH, the defect of HICD.

Again,

Again. From the leſſer Parallelogram A b c D, let there be taken away the Parallelogram h i c D, ſimilar to AEFG, and ſituate alike.

Now, becauſe F D is a Diameter, in the Parallelogram GFKD, G i is equal to i K. - - - 19. 1.
Add h i c D, on both Sides; and G c = h K - Ax. 6. 1.
But, G b = G c. (Th. 1.) wh. G b = h K - - 3. 1.
Add G i, to both; and A i = G K — F i,
conſequently, AF (equal GK) is greater than A i.

Therefore, the Parallelogram AEFG, deſcribed on half the Line, AD, is greater than any other Parallelogram deſcribed on the whole Line; being deficient by a Parallelogram, ſimilar and alike ſituated to that which is deſcribed on half the Line. Q. E. D.

Few, if any, who have favoured the World with Treatiſes on Geometry, have taken notice of this Theorem, or the foregoing, (the 27 and the 24 of Euclid) except thoſe who have trod in his path without ſteping the leaſt aſide; indeed, it is ſo very obſcurely worded, that it is ſcarce intelligible; which, Mr. Stone has endeavoured to remedy, with little ſucceſs. He excuſes Euclid, by ſaying, that he could not have rendered it more clear, in ſo few words; and therefore, rather than appear tedious, gave it as it is; If the Propoſition, itſelf, be unintelligible, how are we to underſtand the Premiſes? Euclid has given ſome Propoſitions in more words. I am perſuaded that I have made it clearer than Mr. Stone, and in as few words as Euclid makes uſe of.

How far this Theorem may be of uſe in the Mathematics I do not pretend to ſay; but, when it is clearly underſtood, it will be found to contain ſomething extraordinary in it; inſomuch, that I could by no means diſpenſe with the omiſſion of it.

There are two Problems, following after, in Euclid, which are dependant on it; he alſo divides a Line in extreme and mean Ratio by it; in the 30th; it is certainly demonſtrable, when done, but how it is to be performed, in the operation, I cannot deviſe. (See, Prob. 11th. B. 2. or Pr. 35. Pr. Geometry.)

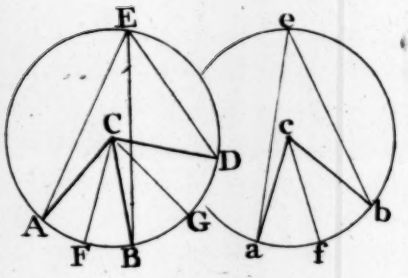
As I do not conceive the Problems to be at all uſeful to Mechanics, &c. I have not inſerted them, amongſt the reſt, in Practical Geometry.

The 32nd Prop. of Euclid is of little conſequence, and has nothing in it worthy of notice.

THE-

THEOREM XIX. 33 Euclid.

In the ſame or equal Circles, the Angles, whether at the Center or at the Circumference, are in proportion to the Arks on which they ſtand.



Firſt, in the Circle AED, the Angle ACB ſtanding on the Ark AB, is to the Angle BCD, on the Ark BD, as AB is to BD.

Let the Arks, AB and BD, be divided into any number of equal Parts, in F and G; and join CF, and CG.

DEM. Now, becauſe the Ark AF=FB, and BG=GD;
the Angle ACF=FCB, and BCG=GCD - C.2.9.3.
wherefore, as the Ark AF:BG::FB:GD } in eq. Ratio } Ax.4.5
and, as the Ang. ACF:BCG::FCB:GCD } in eq. Ratio } Ax.4.5
For, whether AF be equal to, greater, or leſs than BG; the Angle ACF is alſo equal to, greater, or leſs than BCG; in the ſame Ratio. (Th. of Plane Angles.)
conſ. as the Ark AF:BG:: Angle ACF:BCG, &c.
Therefore, as AF + FB : BG + GD,
ſo is the Angle ACF + FCB : BCG + GCD. - 2.5.
i. e. as the Ark AB:BD, :: Angle ACB:BCD. Q.E.D.

But, the Angle ACB, at the Center, is double AEB, at the Circumference; and, BCD is double BED - 9. 3.
wh. as the Angle ACB : BCD :: AEB : BED - Ax. 8.5.
Therefore, as the Ark AB:BD, ſo is the Angle AEB:BED.

2nd. Becauſe, in the equal Circles, AEB, a e b, the Radii are equal; and equal Arks ſubtend equal Angles-C. 2. 9. 3. conſ. as Ark AB:a b :: Angle ACB: a c b; & AEB:a e b.

If AB and a b be divided into an equal number of Parts, the Demonſtration is the ſame as above.

Cor. Sectors of Circles, A C B, B C D, or a c b, are proportional to the Arks AB, BD, or a b.

For, each Sector may be conſidered as an aggregate of a number of Iſoſceles Triangles, of equal Baſes and equal Legs; as ACF, FCB, &c. conſequently, equal Triangles; and having equal Altitudes; each Sector, ACB, BCD, or a c b have, therefore, that Ratio to each other, as the Baſes of thoſe Triangles, i. e. as the Ark AB to BD, &c. - - - - Th. 1.

N. B. None of the following elegant, and ſome of them elementary, Theorems, are in Euclid; except the 34th; which is the 8th Propoſition of his thirteenth Book.

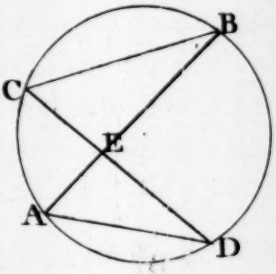
THEOREM XX.

If two Right Lines interſect, within a Circle, and are terminated by the Circumference, the Segments of thoſe Lines are reciprocally proportional.

The two Chords, AB and CD, cut each other in E.

I ſay, that CE is to AE, as EB is to ED.

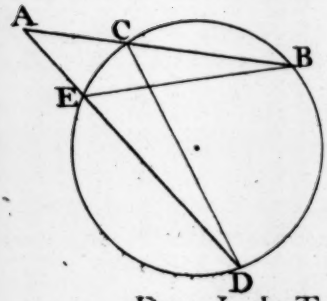
Join AD and CB.



Dem. Now, the Angles AED, CEB are equal. - - 2. 1.
and ABC = ADC; alſo BAD = BCD. - - 10. 3.
wherefore, the Triangles AED, CEB, are ſimilar. - Th. 4.
Therefore, CE:AE::EB:ED, by the ſame. Q. E. D.
Conſ. CE × ED = AE × EB; by 9. 6; alſo by 14. 3.

THEOREM XXI.

If two Right Lines are drawn, from the ſame Point without a Circle, to the oppoſite and concave part of the Circumference; they will have that proportion to each other, reciprocally, as their external Segments.



From the point A, draw AB and AD, at pleaſure, cuting the Circle EBD in the Points E, C, B, and D.

I ſay, that AB is to AD, as AE is to AC. Join EB and CD.

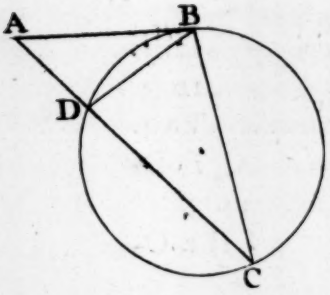
DEM. In the Tri. ABE, ADC, the Angle ABE=ADC - 10.3 and the Angle A is common; wh. AEB=ACD - 10.1. conſ. the Triangles ABE, ADC are ſimilar.

Wherefore, AB : AD :: AE : AC. - - - Th. 4.

And, AB×AC=AD×AE, by Th. 9. and, alſo by 16. 3.

THEOREM XXII.

If from any Point without a Circle, two Right Lines are drawn, one touching the Circle, and the other cuting it; that which touches the Circle is a mean Proportional, between the whole Secant and the external Segment.



From any Point, A, draw A B, touching the Circle at B.

And, from the ſame Point draw AC, at pleaſure, cuting the Circle, in D and C.

I ſay, that AC is to AB, as AB to AD.

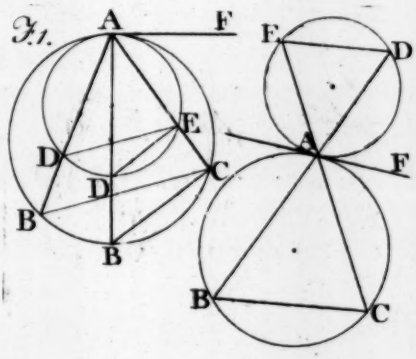
Join BC, and BD.

DEM.

DEM. The Triangles ACB, ABD, are ſimilar. - Th. 4.
For, the Angle ACB = ABD; - - - 13. 3.
and BAC is common; wherefore, ADB = ABC - 10. 1.
Therefore AC : AB :: AB : AD. Q. E. D.
And conſ. AC × AD = AB □. by Cor. Th. 9; alſo by 16. 3.

THEOREM XXIII.

If two Circles touch one another, either internally or externally, and if two Right Lines be drawn, from or through the point of Contact, cuting both Circles; thoſe Lines will be cut proportionally, by the Circumferences of the Circles.



In the Circles, ABC, ADE, and from, or through, the Point of Contact, A, draw AB and AC, cuting both Circumferences, in the Points B, C, D, and E.

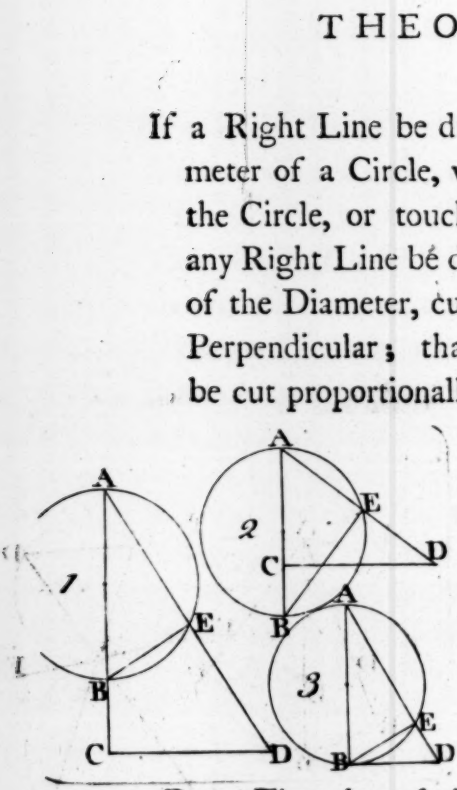
I ſay, that AB is to AC, as AD to AE.

Draw AF, a Tangent to the Circles, at the Point A; and join BC, and DE.

DEM. Then, the Angle ABC is equal to ADE; - Ax. 3. 1.
For they are each equal to the Angle FAC. - Th. 13. 3.
And the Angle BAC (Fig. 1.) is common;
In the ſecond, BAC, DAE, are vertical, th. equal - 2. 1.
therefore, the Triangles, ABC, ADE, are ſimilar,
and, conſequently, AB : AD :: AC : AE. - - Th. 4.

THEOREM XXIV.

If a Right Line be drawn perpendicular to the Diameter of a Circle, whether it be within or without the Circle, or touching the Circumference, and if any Right Line be drawn, from the fartheſt extreme of the Diameter, cuting the Circumference and the Perpendicular; that Line and the Diameter will be cut proportionally.



Let the Right Line CD be perpendicular to the Diameter AB.

Draw AD, at pleaſure, cuting the Circumference, in E, and the Perpendicular, in the Point D.

I ſay, that AB is to AE, as AD to AC. Join EB.

DEM. Then, becauſe A B is a Diameter, the Angle AEB, (being in a Semicircle) is a Right one; - - - 12. 3. and ACD is a Right one (Con.) th. equal to AEB - Ax 9. and, the Angle DAC is common; conſ. ADC = ABE - 10. 1. wherefore, the Triangles AEB, ACD are ſimilar. - 4. 6. Therefore, as AB : AE :: AD : AC. Q. E. D.

In Fig. 3. DB and AC are the ſame; conſequently, AB is a mean Proportional between AE and AD.

2 COR.

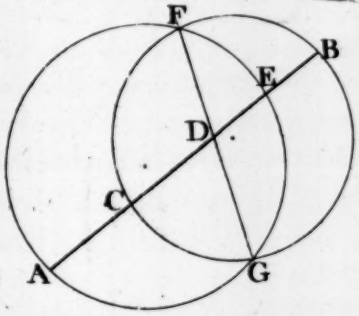
Cor. The Rectangle under any Line, AD, drawn from the extreme A, of the Diameter, to the Perpendicular, DC, and the part of it, AE, within the Circle, is equal to the Rectangle under the Diameter, AB, and the whole, AC, of the Diameter produced to the Perpendicular, (Fig. 1.) For, AE:AB::AC:AD; th. they are four proportional Lines conſequently, AE × AD = AB × AC. - - Th. 9.

N. B. In Fig. 2nd, the Rectangle under the Means, AB and AC, is under the whole Diameter and a part of it.

In Fig. 3rd, ſince AE:AB::AB:AD;
conſequently, the Rectangle under AD and AE, is equal to the Square of the Diameter, AB. - Cor. to 9.
For, AD × ED = BD □ (16. 3.) } Therefore;
and, AB □ + BD □ = AD □ (20. 1.) } as above.

THEOREM XXV.

If two Circles cut each other, and a Right Line be drawn cuting both Circles, it will be cut proportionally, by the Circumferences, and a Right Line joining the points of interſection of the Circles.



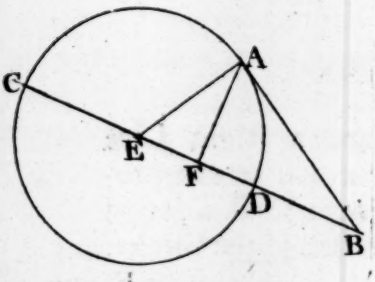
Join the Points, F and G, in which the two Circles, AFG and FBG, cut each other; and, let any Right Line, AB, cut both Circles, and the Right Line FG, in the Points A, C, D, E, and B.

I ſay, the Line AB is cut proportionally in thoſe Points; viz. as AC:CD::BE:ED.

Dem. For, in the Circle AFG, as AD:DF::DG:DE }
And, in the Circle FBG, as DF:CD::DB:DG } - 20.
wherefore, - - as AD:CD::DB:DE - 10. 5.
Therefore, as AD−CD:CD::DB−DE:DE - 7. 5.
That is, as AC:CD::EB:ED. Q. E. D.

THEOREM XXVI.

If a Tangent to a Circle cuts any Diameter produced, and a Perpendicular be drawn, from the Point of Contact to that Diameter; the Segments of the Diameter, intercepted between the Center and the Perpendicular, the Circumference, and the Tangent, are proportionals, i. e. the Radius is a mean Proportional betwixt the part, intercepted between the Center and the Perpendicular, and the Point in which the Tangent cuts the Diameter.



Let any Tangent, AB, cut the Diameter, CD produced, in B; and, let AF be perpendicular to CD; and E the Center.

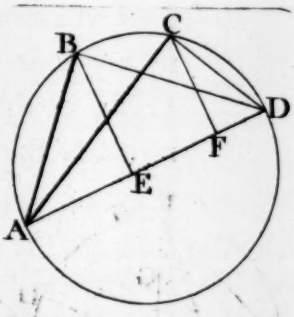
I ſay, that EF, ED, and EB are in continual Proportion.

DEM. For, becauſe EAB is a Right Angle, - C. 3. 8. 3.
and, AF is perpendicular to EB; - - Con.
the Triangles EAF, EAB are ſimilar. - - - Th. 7.
Wherefore, as EF : EA :: EA : EB. - - Cor. 2. 7.
But, EA is equal to ED;
therefore, - as EF : ED :: ED : EB. - - Ax. 4. 5.

THEO-

THEOREM XXVII.

The Squares of Chord Lines, drawn from one extreme of a Diameter, are in proportion to the Parts of the Diameter, intercepted between the Perpendiculars, from the other extremes of the Chords, to the Diameter.



AB and AC are two Chords, having a common extreme, A.

Draw a Diameter, AD, from the Point A; and, from the other Extremes, B and C, of the Chords, draw the Perpendiculars BE and CF, to the Diameter, cuting it in E and F.

I ſay, the Square of AB is to AC ſquare, as AE to AF.

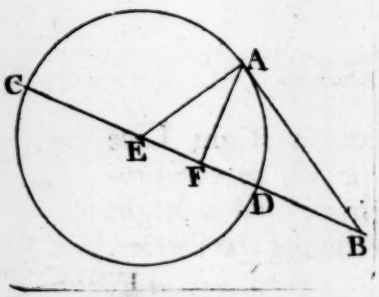
Draw BD and CD.

Dem. Now, BE is a mean Proportional between AE & ED;
alſo, CF is a Mean, between AF and FD - Cor. to 7.6.
for ABD and ACD are Right Angles. - - - 12. 3.
Wh. the Rect. AED = BE □; alſo AFD = CF □ - Cor. 9.6.
But AB □ = BE □ + AE □; and AC □ = CF □ + AF □,
conſ. AB □ = AED + AE □; and AC □ = AFD + AF □.
But, AED, i. e. AE × ED + AE □ = AE × AD; } 3.2.
and AFD, i. e. AF × FD + AF □ = AF × AD; }
wh. AB □ = AE × AD; and AC □ = AF × AD - Ax. 3. 1.
But, AE × AD : AF × AD : : AE : AF. - - Th. 1.
Therefore, AB □ : AC □ : : AE : AF. Q. E. D.

THEO-

THEOREM XXVI.

If a Tangent to a Circle cuts any Diameter produced, and a Perpendicular be drawn, from the Point of Contact to that Diameter; the Segments of the Diameter, intercepted between the Center and the Perpendicular, the Circumference, and the Tangent, are proportionals, i. e. the Radius is a mean Proportional betwixt the part, intercepted between the Center and the Perpendicular, and the Point in which the Tangent cuts the Diameter.



Let any Tangent, AB, cut the Diameter, CD produced, in B; and, let AF be perpendicular to CD; and E the Center.

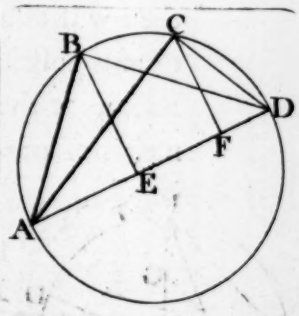
I ſay, that EF, ED, and EB are in continual Proportion.

DEM. For, becauſe EAB is a Right Angle, - C. 3. 8. 3.
and, AF is perpendicular to EB; - - Con.
the Triangles EAF, EAB are ſimilar. - - - Th. 7.
Wherefore, as EF : EA :: EA : EB. - - Cor. 2. 7.
But, EA is equal to ED;
therefore, - as EF : ED :: ED : EB. - - Ax. 4. 5.

THEO-

THEOREM XXVII.

The Squares of Chord Lines, drawn from one extreme of a Diameter, are in proportion to the Parts of the Diameter, intercepted between the Perpendiculars, from the other extremes of the Chords, to the Diameter.



AB and AC are two Chords, having a common extreme, A.

Draw a Diameter, AD, from the Point A; and, from the other Extremes, B and C, of the Chords, draw the Perpendiculars BE and CF, to the Diameter, cuting it in E and F.

I ſay, the Square of AB is to AC ſquare, as AE to AF.

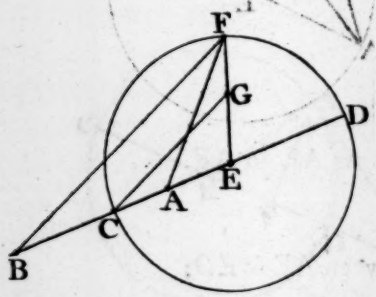
Draw BD and CD.

DEM. Now, BE is a mean Proportional between AE & ED; alſo, CF is a Mean, between AF and FD - Cor. to 7.6. for ABD and ACD are Right Angles. - - - 12. 3.
Wh. the Rect. AED = BE □; alſo AFD = CF □ - Cor. 9.6.
But AB □ = BE □ + AE □; and AC □ = CF □ + AF □,
conſ. AB □ = AED + AE □; and AC □ = AFD + AF □.
But, AED, i. e. AE × ED + AE □ = AE × AD;
and AFD, i. e. AF × FD + AF □ = AF × AD; } 3. 2.
wh. AB □ = AE × AD; and AC □ = AF × AD - Ax. 3. 1.
But, AE × AD : AF × AD : : AE : AF. - - Th. 1.
Therefore, AB □ : AC □ : : AE : AF. Q. E. D.

THEO-

THEOREM XXVIII.

If any Point, except the Center, be taken in a Diameter of a Circle, and if another Point (in the ſame Diameter, produced, and on the ſame ſide of the Center) be taken; the Diſtance of which, from the Center, is a third Proportional, to the diſtance between the Center and the firſt aſſumed Point, and the Radius; then will two Right Lines, drawn from thoſe Points to any point in the Circumference, be in the ſame Ratio, as the internal and external adjoining Segments of the Diameter.



Aſſume any Point, A, in the Diameter CD; in which Diameter, produced, take the Point B (from the Center, E) a third Proportional to EA and EC (Pr. 31.) and, to any Point, F, in the Circumference, let two Right Lines, AF and BF, be drawn.

I ſay, that AF is to FB, as AC to CB.

DEM. For, EA:ED (eq. EC) ::ED:EB - - - - Con.
wherefore, EA : EA + ED :: ED : ED + EB. - 6. 5.
i. e. EA:AD::ED:BD; conſ. as EA:ED::AD:BD - 4. 5.
Now, EA : ED :: ED : EB; and EF = ED;
wh. EA : EF :: EF : EB. - - - - - Ax. 4. 5.
therefore, the Triangles EFA, EFB are ſimilar - Th. 5. 6.
for, the Angle FEB is common, & the Sides are proportional.
Conſ. AF:BF::EA:EF; i. e. ::EA:ED, or as AD:BD;
that is, as AC to CB. Q. E. D.
For, EA:EC (eq ED)::EC:EB (i. e. as AD to BD) - Con.
conſequently, EA : EC :: EC − EA : EB − EC;
that is, - as EA : EC :: AC : CB. - - - - - 3. 5.

Otherwiſe:

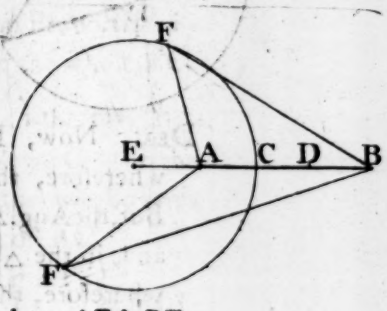
Otherwiſe:

Make EG equal EA, and draw CG.

Then, becauſe EA, EF, are equal EG, EC, reſpectively, and the Angle at E is common, the Triangles AFE, ECG are congruous; and CG = AF. - - - 8. 1.
But, EA:EC::EC:EB (Hyp.) wh. EG:EC::EF:EB conſ. EG:EC::GF:CB (7.5.) wh. CG is paral. to BF-2.6. and the Triangle ECG is ſimilar to EBF - - - 5.
But, GF=AC (Ax. 7. 1.) for EF=EC; and EG=EA; wherefore, as GF, i. e. AC, : CB, or, as EG : EC, or, EF : EB :: CG, i. e. AF, : BF. Q. E. D.

COROLLARY. A PROBLEM.

If any Right Line, AB, be divided at pleaſure, in C; to deſcribe a Circle; to any Point in the Circumference of which, if Right Lines are drawn, from each extreme of the given Line, they ſhall have the ſame Ratio to each other, as the Segments of the given Line.



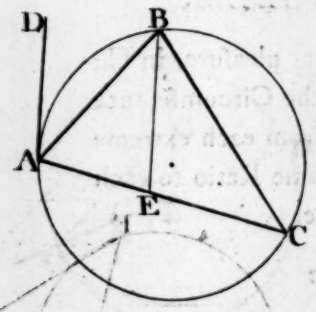
From the greater Segment, CB, take the leſs, AC; i. e. make CD equal AC.
Make AE a third Proportional, to BD and CD (equal AC)
With the Radius EC, on the Center E, deſcribe a Circle;
and to any Point, F, in the Circumference, draw AF & BF.

Then, AF is to BF as AC is to CB. - Theorem.

Becauſe, EA:AC::AC (eq. CD):DB - - Con.
Conſ. EA:EA+AC (i.e. EC) ::AC:AC+DB (i.e. CB)
i. e. as EA:EC::AC:CB. - - Conv. to 6. 5.

THEOREM XXVIII.

If a Triangle be circumſcribed by a Circle, and, a Tangent to the Circle be drawn, at any Angle of the Triangle; and, if a Right Line be drawn from the adjacent Angle, parallel to the Tangent, cuting the oppoſite Side; then, the Side between the Tangent and that Angle, will be a mean Proportional, between the oppoſite Side, and that Segment which is contiguous to the Tangent.

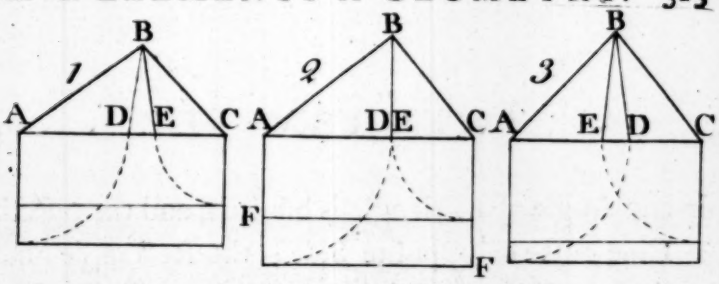


Let AD be a Tangent to the Circle ABCE, at the Point A, (the Angle of a Triangle, ABC, inſcribed) and, let BE be drawn parallel to AD, cuting AC, in E.

I ſay, that AB is a mean Proportional, between AE and AC.

DEM. Now, BE is parallel to AD. - - Con.
wherefore, the Angle ABE = BAD - - - 4. 1.
But, the Ang. ACB = BAD (13 3) conſ. ABE = ACB - Ax. 3. 1
and, in the △s, ABE, ABC, the Angle BAC is common.
wherefore, thoſe Triangles are ſimilar.
Therefore, as AE : AB :: AB : AC. - Th. 4. 6.
conſequently, AE, AB, and AC, are three Proportionals; and therefore, AB is a Mean.

COR. Hence it is manifeſt, that, what is proved, in Cor. 2. of the 7th, in reſpect of a right angled Triangle, is general; in every Scalene Triangle; on any Side of which, ſimilar Triangles are conſtructed. e. g.



In the Triangle ABC, if ABD be made equal to the Angle C, and CBE equal A, the Angles A and C being common; the Triangles ABD, EBC, are ſimilar to the whole Triangle ABC - - - C. 2 4. 6.
and conſequently, to each other. - - Axiom.

Wherefore, the ſquare of AB is equal to a Rectangle, under the Side AC, and the Segment AD;
and, the ſquare of BC is equal to a Rectangle under AC and EC. - - - - - Cor. to Th. 9.

For becauſe the Triangle ABD is ſimilar to ABC;
AD:AB::AB:AC; &, for the ſame reaſon, EC.BC::BC;AC
Wherefore, AB and BC are, each, a mean Proportional; between the adjoining Segments of AC, and the whole, AC.

If the Triangle be right angled (Fig. 2.) the two Segments AD and EC are equal to the whole AC (for the Points D and E coincide; conſequently, the two Squares of AB and BC are equal to the Square of AC.
i. e. AB ſquare is equal to the Rectangle AF, } 20. 1.
and, BC ſquare is equal to the Rectangle FC. - }

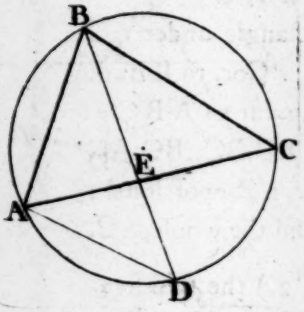
In Fig. 3rd, the Angle at B being acute; conſequently, the two Segments AD and EC are greater than the whole AC; but they are, in each Caſe, ſubject to one general Demonſtration.

When the Angles are all equal, the Square of each Side is equal to the Square of the other; for the Triangle will be equilateral.

THEOREM XXX.

If any Angle of a Triangle is biſected, and the biſecting Line cuts the oppoſite Side; the Rectangle under the two Sides, containing that Angle, is equal to the Square of the biſecting Line, added to the Rectangle under the Segments of the oppoſite Side.

Alſo, the Sides of the Triangle, the biſecting Line, and that Line produced till it cuts the Circumference of a circumſcribing Circle, are Proportionals.



Biſect the Angle B, of the Triangle ABC, by the Right Line BE, cuting AC, in the Segments AE, EC.

Then, the Rectangle under the Sides AB and BC, is equal to the Square of BE, added to the Rectangle under AE and EC.

2nd. About the Triangle ABC, deſcribe a Circle, ABCD. Produce BE to D, and join AD.

I ſay, that AB is to BD as BE is to BC.

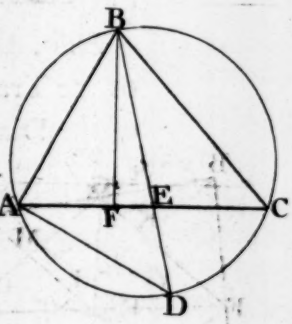
DEM. For, the Triangles ABD, EBC are ſimilar- Cor. 2. 4.
becauſe, the Angles ABD, EBC are equal; - - Con.
and, the Angle ADB is equal to ECB; - - 10. 3.
wherefore, - BAD is equal to BEC. - C. 5. 10. 1.
Th. AB : BD :: BE : BC (4) conſ. AB × BC = BD × BE - 9. 6.
But, □DBE = DEB + EB□ (3. 2.) & DEB = AEC - 14. 3.
Wherefore, the Rectangle DBE = AEC + EB□ - Ax. 6. 1.
Conſequently, AB × BC = EB□ + AE × EC. Q. E. D.

The 2nd Part was proved in the fifth Line.

THE-

THEOREM XXXI.

The Rectangle, under any two Sides of a Triangle, is equal to a Rectangle contained under the Perpendicular to the other Side, from the oppoſite Angle, and the Diameter of a circumſcribing Circle.



Deſcribe the Circle ABCD about the Triangle ABC. - - (4. 4.)
Draw the Diameter BD; and BF perpendicular to AC.

Then, the Rectangle under AB and BC, is equal to the Rectangle under BF and BD. Draw AD.

DEM. Now, the Angle ADB = ACB; - - 10. 3.
and BAD is equal to BFC; - - - - Ax. 9. 1.
(for BFC is a R. Angle (Con.) and BAD is Right - 12. 3)
conſequently, ABD is equal to FBC;
and the Triangles ABD, FBC are ſimilar;
wherefore, as FB : BC :: AB : BD; - - 4. 6.
and therefore, FB×BD=AB×BC. Q. E. D. - 9. 6.

N. B. If the Angle, ABC, was biſected by the R. Line BD; the Rectangle under the Sides, AB, BC, is equal to a Rectangle under the Diameter BD and the Segment BE.

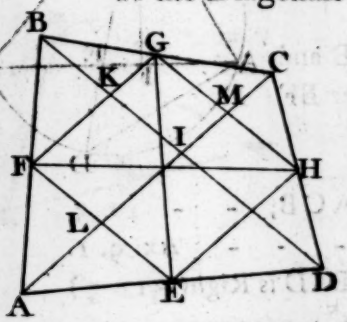
For, AB is to BD, as BE is to BC; by the 30th.

THE-

THEOREM XXXII.

If all the Sides of a Trapezium are biſected, and the Points of biſection, in contiguous Sides, are joined by Right Lines, the Quadrilateral formed by thoſe Lines is a Parallelogram; and, it is equal to half the Trapezium.

Alſo, the Sum of the ſquares of the Diagonals, of the Trapezium, is double the Sum of the ſquares of the Diagonals of the Parallelogram.



Let the Sides of the Trapezium, A B C D, be biſected in the Points E, F, G, and H, and draw EF, FG, GH, and EH.

I ſay, firſt, that the Quadrilateral, E F G H, is a Parallelogram.

DEM. For, becauſe, in the Triangles ABC, ACD, the Sides A B, B C, and alſo A D, D C, are cut proportionally, in F and G, E and H, the Right Lines F G and E H are both parallel to the common Baſe AC; and, for the ſame reaſon, the Lines GH and FE are both parallel to BD; conſequently, they are parallel between themſelves; therefore, E F G H is a Parallelogram. (Def. 33. 1.)

Secondly. It is equal to half the Trapezium A B C D.

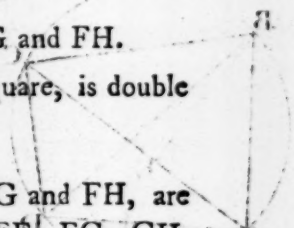
For, in the Triangle A B C, becauſe FG is parallel to AC, and, becauſe AB and BC are biſected in F and G, B I is alſo biſected in K. - - - - 2. 6.

and,

and, for the ſame reaſon AI and I C are biſected in L & M; wherefore, LM is half AC; and conſequently, the Parallelogram L F G M (on half the Baſe A C, of the Triangle ABC, and half its Altitude) is equal to half the Triangle ABC - - - Th. 18. 1. and Pr. 20.
for the ſame reaſon the Parallelogram L E H M = half the Triangle ADC.
conſ. the Par. LG + LH = half the Tri. ABC + ADC,
i. e. the Par. F F G H is equal to half the Trap. A B C D.

Thirdly. Having drawn the Diagonals, EG and FH.

I ſay, that AC ſquare added to BD ſquare, is double the ſum of the Squares, of EG and FH.



For, the Squares of the Diagonals, EG and FH, are equal to the Squares of all the Sides EF, FG, GH, and E H. - - - - - - Th. 14. 2.
But EF and GH are each equal to half BD; and FG, EH, are each equal half AC; } - above.
wherefore, the four Squares, of EF, FG, GH, and EH, are equal to half the Squares of AC and BD - Cor. 4. 2.
and, they are equal to the ſquares of EG and FH -
Therefore, the Squares of AC and BD, are double the Squares of EG and FH.

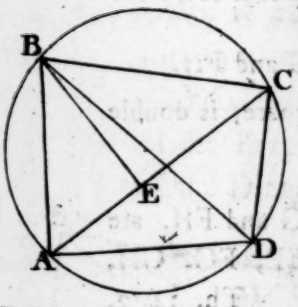
This Propoſition is, by Stone and ſome others, given in the ſecond Book; conſequently, there is a neceſſity for a Lemma, previous to it; which Lemma, is only a particular Caſe of the ſecond Propoſition of the ſixth Book; which cannot be made general without that Theorem.

The third Part of which he makes a ſeparate Propoſition, may be demonſtrated in the ſecond Book; but I did not think it of conſequence enough to make another Propoſition.

THE-

THEOREM XXXIII.

In every Quadrilateral inſcribed in a Circle, the Rectangle contained under the Diagonals, is equal to two Rectangles under the oppoſite Sides.



In a Rectangle the thing is manifeſt - 20. 1.

In the Trapezium ABCD, draw the Diagonals AC and BD.

I ſay, the Rectangle, under AC and BD, is equal to the two Rectangles, under AB and CD, BC and AD.

Make the Angle ABE equal to DBC. (Pr. 4)

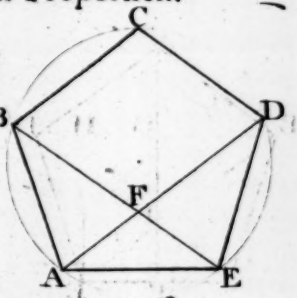
DEM. Now, ABE = DBC (Con) the Angle BAC = BDC - 10. 3
wherefore, the Triangles ABE, DBC are ſimilar - C. 2. 4. 6.
Wh. as AB : BD :: AE : CD (4. 6.) & AB : AE :: BD : CD - 4. 5.
Therefore, AB × CD = BD × AE - - - 9. 6.

Again. The Triangle ABD is ſimilar to EBC.
For, the Angle ABD = EBC (becauſe, ABE = DBC (Con.) and the Angle EBD is common to both.) - Ax. 7. 1.
Alſo, the Ang. BDA = BCA (10. 3.) conſ. BAD = BEC - 10 1.
Wherefore, as BC : BD :: EC : AD - - - 4. 6.
conſequently, BC × AD = BD × EC. - - - 9. 6.
But, - - AB × CD = BD × AE - proved above.
and AE, EC = AC; wh. BD × AE + BD × EC = BD × AC 1. 2
Th. AB × CD + BC × AD = BD × AC. Q. E. D. - Ax. 3. 1.

THEO-

THEOREM XXXIV. 8.13 Euclid.

The Diagonal of a regular Pentagon is to its Side, as a Line divided in extreme and mean Proportion.



In the Pentagon A B C D E, let the two Diagonals, AD and BE, be drawn, cuting each other in F.

Then, A D is to A B, or A E, as A E is to AD − A E.

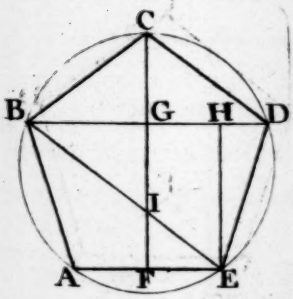
Dem. The Triangles ABE, ADE, are congruous - 8. 1.
For, A B, A E, and E D are equal,
and the Angle B A E = A E D, - - - 8. 4.
wherefore, the Angle A B F = A E F = E A F - 9. 1.
conſ. the Triangles ABE, AFE are ſimilar. - C. 2. 4. 6.
Therefore, as BE : AE : : AE : AF, equal FE. - 4. 6.
But, AE = BF; for DE = BC, and CD = AB - Hyp.
wherefore, BE is parallel to CD, and AD to BC - C. 10. 3.
conſequently, B C D F is a Parallelogram;
wherefore, B F = CD (equal A B, equal A E). - 15. 1.
Therefore, as B E : B F : : B F : F E;
i. e. as B E : A E, or A B, : : A E : F E.
and conſequently, A B, or A E, &c. is equal to the greater Segment of B E, or AD, divided in extreme and mean Proportion.

Cor. 1. The Diagonal of a Pentagon is parallel to the oppoſite Side.

2. Two Diagonals, cuting each other, are mutually divided in extreme and mean Proportion, in the point of interſection.

THEOREM XXXV.

The Diameter or Perpendicular, of a regular Pentagon, is divided in extreme and mean Proportion, by a Diagonal cuting it at Right Angles.



In the Pentagon A B C D E, let the Diameter C F be drawn, and the Diagonal B D, cuting C F in G, perpendicularly.

I ſay, that C F is divided in extreme and mean Proportion, in G.

Draw E H parallel to C F; alſo, the Diagonal BE, cuting the Perpendicular in I.

DEM. Now, becauſe BE is parallel to CD, - C.1. 34.
the Angle C D B is equal to D B E - - - - 4. 1.
and, CBD = CDB (9. 1.) wh. CBD = DBE - Ax. 3. 1.
But, the Angle CGB=BGI (Ax.9) wh. GI=GC - 11.1.
(for the Tri. BCG, BIG are congruous; BG being common)
Now, ſince EH is par. to FG (Con) & AE to BD; by the laſt;
FGHE is a Parallelogram; wherefore, EH=FG - 15.1.
And, becauſe EH is par. to FG; EH:IG::BE:BI - 4.6.
(for the Triangles BGI, BHE are ſimilar; by C. 3. 2. 6)

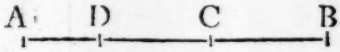
But, BE is to B I, as the greater Segment to the leſs, of a Line divided in extreme and mean Proportion. - 34.
(for, BI=BC) FG=EH; alſo, CG = GI, proved above;
wherefore, FG : GC :: BE : BI; i. e. as BI : IE;
and therefore, CF is divided in extreme and mean Ratio, iu the Point G, where it is cut by the Diagonal BD.

COR. The Diagonal BE, cuts, FG, the greater Segment of CF, in extreme and mean Proportion; at I.

For BD is par. to AE; conſ. the Tri. GBI, IEF are ſimilar;
wherefore, as BI : IE :: GI : IF - - - - - Th. 4.
But, BI:IE::BE:BI(eq.AB) therefore, FG:GI::GI:IF.

Hence, if a Right Line be divided in extreme and mean Proportion, and from either extreme of the greater Segment the meaſure of the leſs is ſet off, it will alſo be divided in the ſame Ratio; and the leſſer Segment of the firſt, will be the greater Segment of the other. e. g.

Let AB be divided in extreme & mean Proportion, in C.



Then, if CD be made equal to CB, AC will be divided in the ſame Ratio; and CD is the greater Segment.

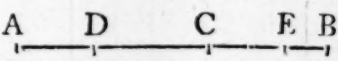
For, ſince AB : AC :: AC : CB, (Hyp.) it will be, as AC:CB::AB−AC:AC−CB, i.e. as AC:CB::CD:AD.

Alſo. If to the whole Line, divided in extreme and mean Proportion, there be added the greater Segment; then, the whole compounded Line is divided in the ſame Ratio, and the greater Segment, of the firſt, will be the leſſer.

Let AC be divided in extreme and mean Proportion, in D. Then, if CB be made equal CD, CB will be the leſſer Segment of the whole, AB, divided in the ſame Ratio.

For, ſince AC : CD :: CD : AD; it will be as AC:CD::AC+CD:CD+AD; i.e. as AC:CD::AB:AC.

Hence, if either Segment be given, the other may be found.



Let AC, the greater Segment, be given.

Divide AC in the Ratio required, in D. - (Pr. 35.)

Produce AC; make CB equal CD,

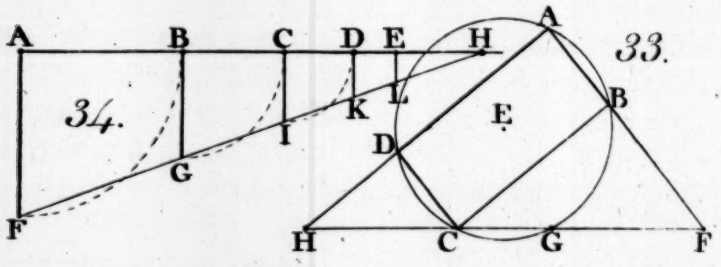
and CB is the Segment ſought.

If CB, the leſſer Segment had been given.

To find the greater; divide CB as before, in E;

make CD equal CB, and AD equal CE.

Then, AC is the greater Segment required.



A Demonſtration of the 33rd Problem.

FG is equal to CH (Con.) conſ. FC is equal to GH - Ax. 6. 1. wherefore, the Rectangles CFG, GHC, are equal - Ax. B. 2. But, AF × BF = CF × FG; alſo, AH × DH = GH × CH - C. 16. 3 conſequently, the Rectangles AFB, AHD, are equal - Ax. 3. 1. Wherefore, as AF : AH :: DH : BF, reciprocally - C. 1. 8. 6. But, as AF : AH :: BF : BC (Th. 4) wh. BF : BC :: DH : BF - Ax. 13. 5

Again. It has been ſhewn (changing Terms) DH : BF :: BF : BC alſo, that BF : BC :: AF : AH ; i. e. as DC : DH. - Th. 4. Therefore, the four Lines, DC, DH, BF, and BC are in continual Ratio ; and conſequently, DH and BF are two Means, between DC and BC ; equal X & Z, by Conſtruction.

A Demonſtration of the 34th Problem.

Firſt, AH : BH :: AB : BC; i. e. as AF : BG - - Th 4.
wherefore, AH : AB :: BH : BC - - - - 4 5.
conſ. AH — AB (eq. BH) : AB :: BH — BC (eq CH) : BC - 7. 5.
wherefore, AB : BH :: BC : CH - - - 5. 5.
conſequently, AB + BH : BH :: BC + CH : CH. - 6. 5.
i. e. AH : BH :: BH : CH ; and as CH : DH, &c.
But, AH : BH :: AF : BG; and, BH : CH :: BG : CI, &c. - 4. 6.
and AH, BH, CH, &c. are in continual Ratio ;
th. AF : BG : CI : DK, i. e. AB : BC : CD : DE, :: AH : BH, &c ÷

But, AH : AF :: BH : BG, &c. and AH is greater than AF.
wherefore, DH is greater than DK, and EH than EL;
conſequently, EL may ſtill be taken from EH, *ad infinitum.*
th. AH is the whole ſum of infinite Proportionals, to X and Z.

ELEMENTS OF GEOMETRY.

BOOK VII. The XI. of Euclid.

THE ſeventh Book, of theſe Elements, treats firſt of the Elements or firſt Principles of plane Solids, (i.e. Solids bounded by, or contained within Planes, only,) viz. of the Poſitions of Right Lines, and of Planes, to Planes; of the Sections of Planes by Planes, and of Right Lines by Planes.

Secondly, of ſolid Angles, generated by the interſections of Planes; their Conſtruction and Principles inveſtigated.

Thirdly, of Solids, contained within ſix parallel Planes, only; their Affections, Proportions, and Properties, are ſearched out, on which is founded the Theory of Menſuration of Solids.

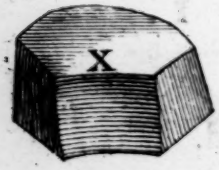
The Doctrine of Solids, contained in this Book and the following, is alſo of great uſe throughout the Mathematics; ſome of which have their very exiſtence in it. The Science of rectilinear Perſpective is wholly founded on the Sections of Planes by Planes, and, of Right Lines cut by a Plane, and by parallel Planes; inſomuch that, without their aſſiſtance, Perſpective would be a very imperfect Science; which, to the immortal Fame of Dr. Brook Taylor, is now eſtabliſhed on the moſt permanent Principles.

Every Theorem, of the eleven firſt, are more or leſs uſeful in Perſpective; the 5th and 7th are of great uſe; which, with the 8th and 9th contain the Eſſence of it, in Theory; as the 10th and 11th, of both Theory and Practice.

To enumerate every branch of the Mathematics, in which this ſeventh Book is particularly uſeful, is needleſs, was it in my power; the Reader may depend on it, that its Doctrine is, at the ſame time, entertaining, and of extenſive utility.

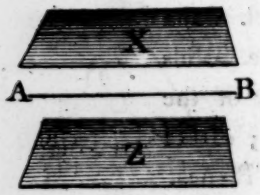
DEFINITIONS.

Def. I. A SOLID is a BODY, having length, breadth, and thickneſs; and is bounded by Planes, or curved Surfaces, or both. As X.



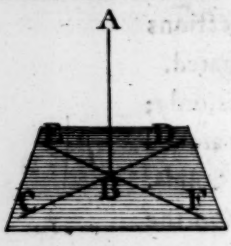
Def. II. A Plane (or Right Line) is parallel to another Plane, which, being produced infinitely, would never meet; but, are every where, equi-diſtant.

As the Plane X, to the Plane Z;
or, the Right Line AB, to both X and Z.



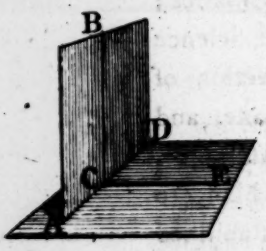
Def. III. A Right Line is perpendicular to a Plane, when it makes equal Angles with every Line drawn in the Plane, to or from that Point in which the Line cuts, or would cut, the Plane.

If AB makes Right Angles with the two Lines, CD and EF, paſſing through B, in which Point, AB cuts the Plane C E D F; and if the Lines, CD and EF, are both in that Plane; AB is perpendicular to the Plane.



Def. IV. A Plane, cuting another Plane, is perpendicular to the other, when it does not incline to the other on either Side.

If the Plane ABD cuts the Plane AED, in AD; and, if any Right Line, B C, drawn in one, be perpendicular to the other Plane (A E D) Thoſe Planes are perpendicular to each other.

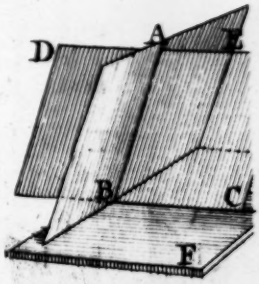


N.B. It is uſual, to define perpendicular Planes, by Right Lines being drawn perpendicular to their common Section; which, I preſume, is taking for granted that their common Section is a Right Line, and which, being granted, there would be no occaſion afterwards to demonſtrate it, in Propoſition 3rd.

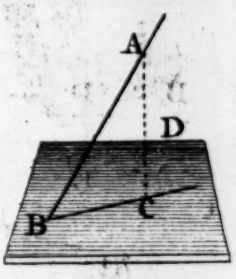
Let the young Geometrician take particular notice, that no regard is had to the poſition of the Plane, or Line, in reſpect of the Horizon, to which another Line or Plane is ſaid to be perpendicular, but only to their poſition in reſpect of each other, for, if they make Right Angles with each other, they are perpendicular, each to the other.

Def. V. If a Plane (or Right Line) be neither parallel nor perpendicular to another Plane, it is ſaid to incline to the other Plane.

The inclination of one Plane to the other is the acute Angle ABC, made by the Section of another Plane, DEC; cuting both the inclined Planes, BE and BF, perpendicularly; i. e. at Right Angles.



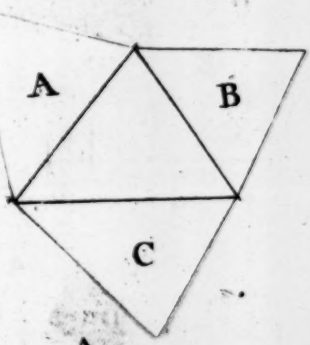
Alſo, the inclination of a Right Line, AB, to a Plane, BCD, is the Angle ABC made by another Line; drawn in the Plane BCD, from the Point B, in which the inclined Line, AB, cuts the Plane; and paſſing through another Point, C, in which a Perpendicular, AC, from any Point, A, in the inclined Line, cuts the Plane.



N. B. Two Planes have an equal inclination, to two other Planes when the Angles of their inclination are equal to one another.

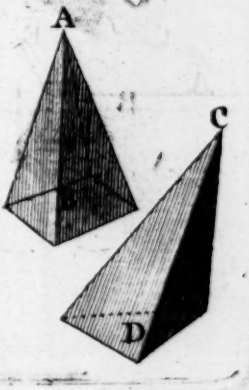
Def. VI. A SOLID ANGLE is that which is made by more than two Plane Angles, applied cloſe to each other, at the ſame Point, ſo, that two of them are not in the ſame Plane.

If the three plane Angles A, B, and C, be turned over, they will form a ſolid Angle, at the Point where their Vertices meet each other.



N. B. All the Plane Angles forming a ſolid Angle are, together, leſs than four Right Angles. (See Theorem 12.)

If three Plane Angles, form a ſolid Angle, any two muſt be greater than the third; or, three, greater than a fourth, &c. (See Theorem 13.)

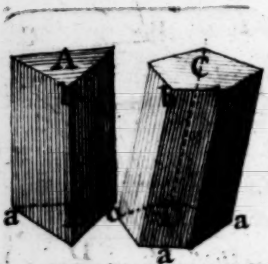


Def. VII. A PYRAMID is a SOLID, contained by any number of Planes more than three; of which, one (the Baſe) may have any number of Sides; all the other Planes are Triangles, meeting in a common Vertex.

As AB, or CD; A and C are their Vertices; B and D are their Baſes.

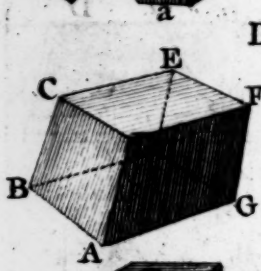
N. B. Not leſs than four Planes can form a Solid; and that, muſt neceſſarily be a Pyramid, and the Planes are all Triangles.

Therefore a Pyramid may be ſaid to be the firſt of Plane Solids; nevertheleſs it is not the ſimpleſt.



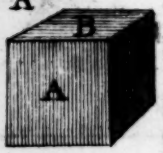
Def. VIII. A PRISM is a SOLID, contained within any number of Planes more than four; of which, two are parallel, equal, and ſimilar Figures (of any number of Sides) all the other Planes are Parallelograms.

As AB; or CD. A and B, C and D are the ſimilar and equal Planes; the others, a b, b c, &c. are all Parallelograms.

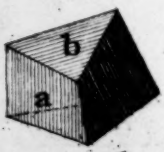


Def. IX. A PARALLELOPIPED is a PRISM, bounded by Six Planes; of which, the oppoſite Planes are parallel.

If the Plane ABCD be parallel to DEFG, ABDG to DCEF, and ADFG to BCED; the Solid, ACF, is a PARALLELOPIPED.



Def. X. A CUBE is a PARALLELOPIPED, whoſe Planes are all Squares. As A, B, C.



Def. XI. SIMILAR PLAIN SOLIDS are contained within an equal number of plane Figures, ſimilarly ſituated and ſimilar one to another, reſpectively, in each Solid.

The Angles are, alſo, neceſſarily equal, each to the other, reſpectively.

If the Plane a, be ſimilar to A, b to B, and c to C; and if all the other Planes, of which each Solid is conſtructed, be alſo ſimilar and ſituate alike; then the Solid a b c is ſimilar to A B C.

All CUBES are ſimilar Solids.

All SOLIDS, contained within an equal number of equilateral and equiangular Planes, are ſimilar.

* A Priſm may be conceived to be generated, by the direct motion of any right lined Plane Figure; always parallel to its firſt poſition, but not in a continuation of the Plane of the Figure.

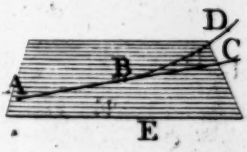
AXIOMS.

Ax. I. The firſt Propoſition of Euclid.

Every part of a Right Line is in the ſame Plane. i. e. One part of a Right Line cannot be in any Plane, and another part of the Line out of that Plane.

For, if it was poſſible, a Right Line does not agree with a Plane in every Point; agreeable to Def. 6. of Book 1ſt.

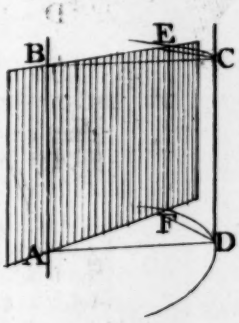
If the part AB, of a Right Line, ABC, be in the Plane AEC, the whole Line is in the ſame Plane; for, the two Points, A and B, are in the Plane; and if any Point, D, be not in the Plane, ABD is not a Right Line; ſeeing that, AB is a Segment of the Right Line AC.



Ax. II. Two Right Lines, being parallel, are in the ſame Plane, i. e. a Plane, may paſs through both Lines.

AB & CD, being parallel, are in the Plane ABCD.

Imagine a Plane, ABEF, revolving on AB as an Axis. Then, ſince CD is parallel to AB, it is parallel to a Plane paſſing through AB; wherefore, the Perpendiculars CE and DF are equal (Def. 2.) and, conſequently, when the Point C coincides with E, D will coincide with F, and the whole Line CD is in the Plane, ABCD.

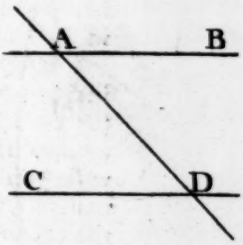


Ax. III. The ſeventh Propoſition of Euclid.

Two Right Lines, being parallel, and cut by another Right Line, are all in the ſame Plane.

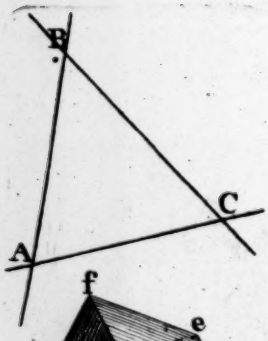
For, if AB be parallel to CD, they are in the Plane ABCD (2nd) conſequently, the Points A and D, where AD cuts them both, are in that Plane.

Wherefore, ſince there are two Points, A and D, of a Right Line AD, in a Plane, the whole Line is in that Plane (Ax. 1.) and, conſequently, AD is in the ſame Plane with the two parallel Lines, AB and CD.



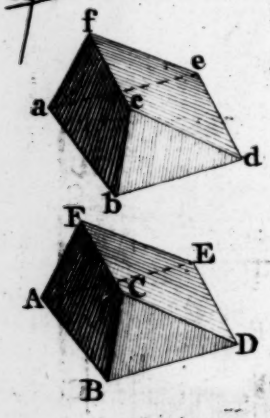
Hence, if two or more Lines, not parallel, are in the ſame Plane, and are both cut by another Right Line, it is alſo in the ſame Plane with them. Conſequently, two Right Lines (AB and AD) cuting each other, are in the ſame Plane.

Ax. IV. The ſecond Propoſition of Euclid.



Three Right Lines cuting each other (not in the ſame Point) are all in the ſame Plane; wherefore, the three Sides of every right lined Triangle are in one Plane.

For, the two Right Lines, AB and BC, are in the ſame Plane (3.) Conſ. being both cut by another Right Line, (AC) ſeeing, the two Points, A and C, are in the ſame Plane, the whole Line, AC, is alſo in that Plane - Ax. 1.



Ax. V. Solids, contained within an equal number of Planes, which are equal and ſimilar to each other, two and two, (i. e. one in each Solid) reſpectively; and being ſimilarly ſituated, in reſpect of each other, and alſo of the whole; ſuch Solids are equal in every reſpect. As A C E and a c e.

If the Plane a b d e, be equal and ſimilar to A B D E; a c, to AC, and c e to C E; alſo a f e, to A F E, and b c d, to B C D; then, the Solid a b c d e f, is equal in every reſpect, to A B C D E F.

The eleventh Book of Elements is, in general, leſs underſtood than any of the foregoing; not owing to any deficiency or imperfection in the Demonſtrations, but very much ſo to the imperfection of the Diagrams. It very rarely happens (and which never, yet, has happened) that a Mathematician, who had engaged in a Publication of the Elements of Geometry, was alſo, a competent Draftſman, in Perſpective; for want of which, the Schemes, which are intended to repreſent Planes cuting Planes, &c. are ſo badly deviſed, and worſe repreſented, that it is with the greateſt difficulty a juſt Idea of their meaning can be communicated; nor is it poſſible, unleſs the Pupil has very acute Talents, or is aſſiſted in his ſtudy of them, by ſome Apparatus, or an able Tutor.

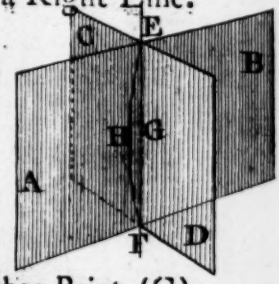
I flatter myſelf that I have removed that difficulty, which has hitherto been the greateſt impediment, to a clear inveſtigation of the Doctrine of Solids. Where it is neceſſary, moveable Schemes are adapted for the purpoſe; and, where they may be diſpenſed with, the Diagrams are juſtly, and perſpectively, repreſented; ſo that, the Student muſt not always expect (as in Plane Geometry) that, Angles which are ſaid to be right, or Lines parallel, &c. are really ſo, in the Diagram; but, that they are, or would be ſo, if ſuch a Figure was really conſtructed, as the Diagram repreſents. For, unleſs the Reader can conceive this, it will be very difficult for him to ſuppoſe one Line to be greater than another, when it is, abſolutely and evidently, leſs; which, from the Premiſes given and the foregoing Elements, it is manifeſt muſt be greater.

Therefore it is, in ſome meaſure, neceſſary (by way of Poſtulate) to requeſt, that one Line be equal to another; or at right angles, i. e. perpendicular to another Line, or Plane, &c; that two Right Lines contain an Angle, equal to that which two other Lines contain; which, in the Diagram, is, perhaps, either greater or leſs. Theſe things muſt be granted and underſtood to be ſo, or no Demonſtration, of what is intended, can poſſibly follow.

I thought it neceſſary to appriſe the young Student of theſe preliminaries; otherwiſe (without a Tutor) he would, frequently, be perplexed and bewildered; and would, very probably, conclude, that there muſt be ſome miſtake, or error in the Diagram; for it is not eaſy, at firſt, to conceive, that one Line ſhould be perpendicular to another, when we ſee and know, that the Angles they make, are not Right ones; and, being acute, that another Angle, apparently contained within the other, and manifeſtly leſs, is a Right Angle; i. e. underſtood to be ſo.

THEOREM I. 3 Euclid.

The common Section of two Planes is a Right Line.



Let the two Planes, AB and CD, cut each other, in EF.

I ſay, that EF is a Right Line.

Dem. The Points E and F, and every other Point (G) in the Line EF, are in both Planes; for, the Line FG is in both Planes; ſeeing it is their common Section.-Ax. 1. Therefore, it is a Right Line; ſeeing that, a Plane agrees with a Right Line in every Point; - - Def. 6. 1. and, a curved Line cannot poſſibly be in two Planes.

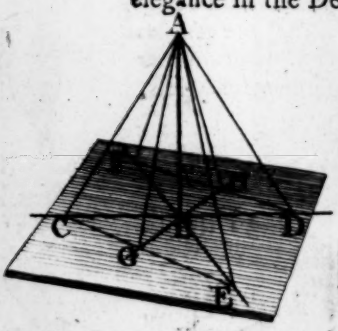
For, ſuppoſe the curve Line, EHF, drawn through the Points E and F, in the Plane ABCD; if it was their common Section, it muſt alſo be in the Plane, CD; and conſequently it would not be a Plane, but a convex Surface.

This Theorem might paſs for an Axiom, being ſelf-evident.

THEOREM II. 4 Euclid.

If a Right Line ſtands at right angles, in the Point of interſection of two Right Lines; it will be perpendicular to the Plane, in which thoſe Lines are.

This might alſo paſs for an Axiom; but there is a peculiar elegance in the Demonſtration, which cannot be diſpenſed with.



Imagine the Right Line, AB, ſtanding at right angles to the two Right Lines, CD and EF, in the Point B of their common Section.

Then, AB will be perpendicular to the Plane CEDF, in which the two Lines, CD and EF, are ſituated.

Make BC equal to BD (at pleaſure) and imagine BE, BF, alſo equal to BC, BD.

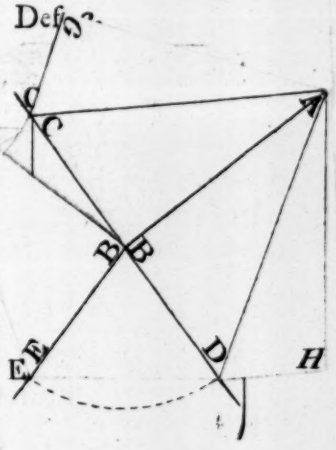
AB ſtanding perpendicular to them, imagine the Right Lines AC, AD, AE, and AF, drawn to any Point, A, in AB.

Join CE and DF; and, through B, draw any Right Line, GH, at pleaſure, cuting CE and FD, at G and H; and draw AG, AH.

DEM. Then, becauſe CB, BD, BE, and BF are all equal, and AB is common; alſo, the Angles ABC, ABD, ABE, ABF, are all ſuppoſed Right, therefore equal; the Right Lines AC, AD, AE, and AF, are all equal. - - 8. 1. wh. the Angle ACE=AEC; and ADF=AFD - 9. 1. And, becauſe the Angle CBE=DBF (2. 1.) and CB, BE are equal reſpectively, to BD, BF; CE=FD - 8. 1. conſequently, the Angle BCE=BDF, - ſame. Alſo, the Triangle CAE, being equilateral to FAD*, they are conſ. equiangular; & the Angle ACE=ADF-7. 1.

Now, the Angle BCE = BDF; and CBG = DBH - 2. 1.
and CB = BD; wh. the Triangle CBG = DBH; } - 11. 1.
and, conſequently, CG = DH, and BG = BH. }
Wherefore, in the Triangles CAG, DAH; AC, CG, are reſpectively equal to AD, DH, and contain equal Angles; conſequently, AG is equal to AH. - - 8. 1.
wh. the Triangles, AGB, BAH, are equilateral, to each other; (for BG, was proved, equal to BH, AG equal AH, and AB is common) conſequently, the Angle ABG = ABH.
Therefore, AB is perpendicular to GH - Def. 10. 1.
and, conſequently, to every Right Line, drawn through B, in the Plane CEDF; and therefore, AB is perpendicular to the Plane, CEDF. Q. E. D. - - Def. 3.

To aſſiſt the Imagination, I have added, Fig. 2. in which, if the Plane CAD be turned up (on CD) perpend. and, the Triangle ABE alſo vertical, till AB, in or coincides with AB in the other; then, turning over Triangles CA*G*, DA*H*, till *G* and *H* fall on G and and turning back the Triangle AB*F*, till its Baſe fa into the Right Line EF; in which poſition may b ſeen, what is repreſented in Figure 1.



The Line AB is perpendicular to the Plane CEDF.
For, CB, BE, BD, and BF are equal; and conſequently AC, AE, AD, and AF are equal; alſo, AG is equal to AH, and the Angles ABG, ABH are Right; conſequently AB is perpendicular to the Plane CEDF.

COROLLARY. Propoſition V. of Euclid.

If three Right Lines meet in one Point, and a fourth ſtands at right angles to each, in that Point, the three Lines are in the ſame Plane.

If AB makes Right Angles with CB, and BE, and alſo with BG, they are neceſſarily in the ſame Plane - Def. 3.

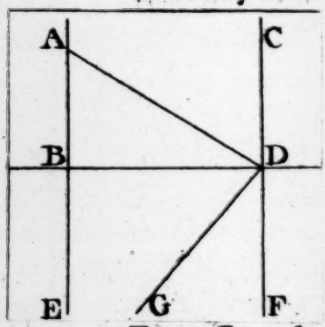
For, if ABG be leſs than a Right Angle, BG will be on this ſide the Plane, CHF; and, if it exceeds a Right Angle, BG will fall on the other Side. But, ABG is a Right Angle. Therefore, BG is in the ſame Plane with BC and BE.

* Triangles are equilateral to one another, when the Sides of one are equal, reſpectively, to the Sides of the other.

THEOREM III. 6 Euclid.

If two Right Lines be perpendicular to the ſame Plane, they are parallel to one another.

This might well paſs for an Axiom; for, it is evident, the two Lines may be in the ſame Plane, which is thus proved.



Let AB and CD be at right angles with the Plane B E F D.

I ſay, that AB is parallel to CD.

Draw BD, through the points, B and D, in which the Lines, AB & CD, cut the Plane. Alſo, draw BE and DF, perpend. to BD.

DEM. Becauſe AB and CD are both perpendicular to the Plane BGD, they are at right angles with every Line in that Plane, drawn through the Points B & D, reſpectively; therefore, to BE and DF (2.) and, conſequently, they don't incline to the Plane BGD, on any Side - Def. 3.
Wherefore, a Plane may paſs through both Lines - Def. 4.
But, the Angles ABD and CDB are Right - Def. 3.
Therefore, AB is parallel to CD - - - Th. 4. 1.

If the Plane BACD be turned up perpendicular, to BGD, the thing is manifeſt; ſeeing, the Lines AB and CD, make Right Angles with BE and DF, reſpectively.

According to Euclid (AD being drawn) the three Lines, AD, BD, and CD, are proved to be at right Angles with DG, ſuppoſed perpendicular to BD; but, the proceſs, though ſtrictly true, appears very lame and contradictory in the Figure.

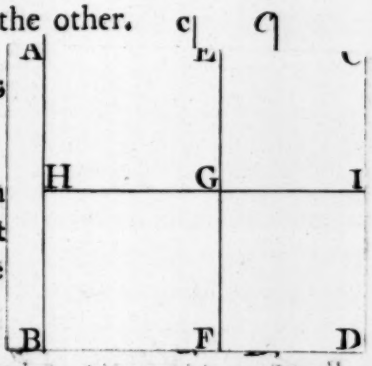
COROLLARY. The 8th Propoſition of Euclid.

If two Right Lines be parallel, and if one of them be at right angles with ſome Plane, the other Line is alſo at right angles with the ſame Plane.

This, being the Converſe of the Theorem, is manifeſt.

THEOREM IV. 9 Euclid.

Right Lines being parallel to the ſame Right Line, which is not in the ſame Plane with them, are parallel amongſt themſelves; i. e. each to the other.



Let AB and CD be each parallel to EF; AB is parallel to CD.

In EF, take any Point, G, from which draw Right Lines, GH and GI, both at right angles with EF, and cuting the Lines AB and CD reſpectively.

Dem. By the Premiſes, EF is not in the ſame Plane with AB and CD; therefore, raiſe up EF, out of the Plane, but parallel to it, till CD coincides with *CD*, or any other Line, c d, parallel to EF.

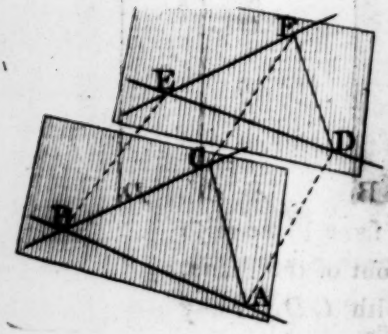
Then, becauſe GH and GI are at right angles with EF, EF is perpend. to a Plane paſſing through thoſe Lines - Th. 2
But AB and CD are both parallel to EF;
wherefore, they are alſo at right angles with the Plane HGI, paſſing through GH and GI - Cor. to Th. 3.
Therefore, they are parallel between themſelves; by Th. 3.

When the three Lines are all in the ſame Plane, as in Theo. 5. Book 1ſt. it is manifeſtly an Axiom; and, in this Caſe, it amounts to little more; ſeeing that, a Plane may paſs through any two Right Lines, which are Parallel to one another. - - Ax. 2.

THE-

THEOREM V. 10 Euclid.

If two Right Lines, cuting or meeting each other, be reſpectively parallel to two other Right Lines, alſo meeting or cuting one another, though not in the ſame Plane with the firſt, they ſhall contain equal Angles.



Let the two Right Lines, AB and BC, be reſpectively parallel to DE and EF; viz. AB to DE, and BC to EF; and, ſuppoſe they are not in the ſame Plane.

I ſay, the Angle ABC is equal to DEF.

Take BA equal to ED, BC equal to EF, and draw the Right Lines AD, BE, & CF.

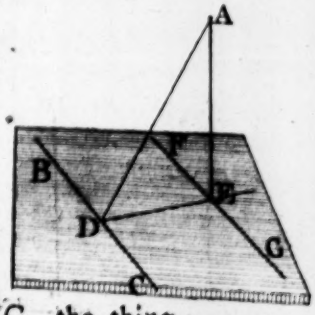
Dem. Becauſe AB is equal and parallel to DE, - Hyp.
AD is equal and parallel to BE - - Cor. to 15. 1.
and, for the ſame reaſon, BE is equal and parallel to CF;
wherefore, CF is equal and parallel to AD - Ax. 3. 1 & Th. 4.
Join AC and DF, by Right Lines.
Now, becauſe AD is equal and parallel to CF,
AC is equal (and alſo parallel) to DF; as above.
Then, in the Triangles ABC, DEF, the three Sides, AB, BC, and AC, of the one, are reſpectively equal to DE, EF, and DF of the other;
wherefore, the Angles, of one, are alſo, reſpectively, equal to the Angles of the other. - - - - 7. 1.
Therefore, the Angle ABC is equal to DEF;
being oppoſite equal Sides, AC and DF. Q. E. D.

From the foregoing Theorems may be deduced the following Problems.

PROB-

PROBLEM I. 11 Euclid.

From any given Point, to draw a Right Line perpendicular to a Plane, in which that Point is not situated.*



Let A be the given Point, and BEC a Plane.

In the Plane BEC, draw, at pleasure, the Right Line BC, and, from A, draw the Right Line AD, perpendicular to BC. - - - (Prob. 7.)

If AD be perpendicular to the Plane BEC, the thing is already done; but, if it be not, proceed as follows.

From the Point D, draw DE, in the Plane BEC indefinite, also perpendicular to BC; (Prob. 6.) and from A, draw AE perpendicular to DE. (Prob. 7.)

I say, that AE is perpendicular to the Plane BEC.

Through the Point E, draw FG, parallel to BC; (Pr. 5.) consequently, it is in the same Plane with BC. - Ax. 2. because the Point E is in the Plane.

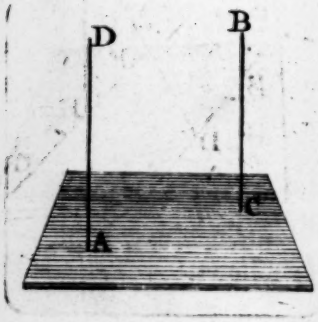
DEM. Now, because AD & DE are both perpendicular to BC, BC is perp. to a Plane passing through AD & DE - Th. 2.
But, FG is parallel to BC; by Construction;
wherefore, FG is also perpendicular to the Plane ADE.
Now, since AE is perpendicular to FG, and, also to DE, it is perp. to the Plane in which those Lines are situated - 2.
Therefore, AE is perpendicular to the Plane BEC. Q.E.D.

---

* Professor Simson says, from a Point given above the Plane. I am not a little surprized at the expression, from one of his extraordinary Sagacity; because, the same thing holds true however the Plane be situated; or the Point in respect of the Plane, provided it be not in the Plane.

PROBLEM II. 12 Euclid.

From a Point given in a Plane, to draw a Right Line perpendicular to the Plane.



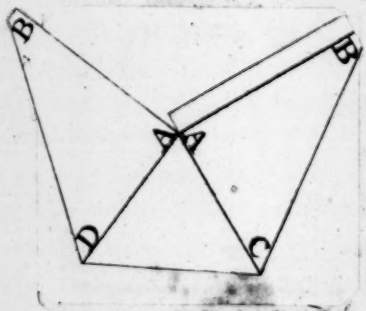
Let A be the given Point, in the Plane AC.

Aſſume any Point, B, out of the Plane; and, from that Point, draw BC perpendicular to the Plane; by the foregoing.

Then, from the Point A, draw AD, parallel to BC. - - (by Prob. 5. 1.) AD is perpendicular to the Plane AC.

DEM. Becauſe BC is perpendicular to the Plane - (Prob. 1.) and AD is parallel to BC; by Conſtruction; AD is perpendicular to the Plane AC. - - Th. 3.

I believe, that the manual operation, of this Problem, would be eaſier performed by two Right Angles. Thus



Let A be the given Point, at which a Perpendicular is required, to the Plane ACD.

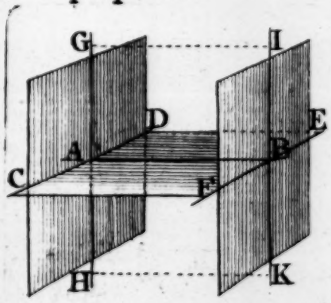
Apply a Right Angle, BAC, to the Point A, at pleaſure, on AC; and, in any Angle (CAD) apply another R. Angle BAD. i. e. raiſe up the two right angled Triangles BAC, BAD, on AC and AD, till the Points, B, coincide. Then, AB is perpendicular to the Plane CAD. - Th. 2.

COR. Hence it is manifeſt, that there cannot be drawn two Lines, from the ſame Point, perpendicular to a Plane, and on the ſame Side.

THEO-

THEOREM VI. 14 Euclid.

Planes, to which the ſame Right Line is perpendicular, are parallel to one another.



Let the Right Line AB be perpendicular to both the Planes, CD and FE.

I ſay, thoſe Planes are parallel.

Imagine a Plane, CDEF, to paſs through the Line AB, cuting both Planes, in CD, and EF; which Lines, neceſſarily paſs through the Points A and B, in which the Line AB cuts thoſe Planes, reſpectively. - Ax. 1. and Th. 1.

At the Points A and B, draw AG and BI, in the Planes CGDH and FIEK, perpendicular to CD and EF (Prob. 6.) and produce them to H and K.

Dem. Then, becauſe GH and IK are perpendicular to CD and EF, and theſe Lines are in the Plane CDEF; GH and IK are perpendicular to that Plane; - Th. 2. (Becauſe, AB being perpendicular to both the Planes, CGDH and FIEK, it is perpendicular to every Line drawn through the Points A and B; th. to GH & IK) wherefore, they are parallel to one another; - Th. 3. and, conſequently, the two Planes cannot meet; being produced, either way, towards G and I, H and K - Ax. 1.

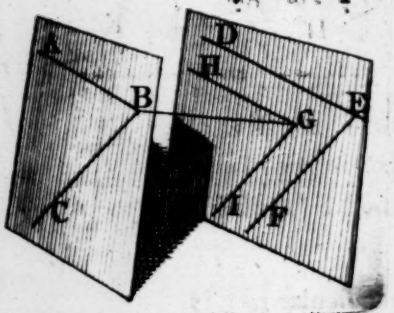
Again. Draw any other Right Line through A & B, in the Planes CGDH and FIEK, at right angles, with, or making equal Angles, and inclined the ſame way towards GH and IK; as CD and FE.

Then, becauſe the Angle GAD = IBE, and AG is parallel to IB; and thoſe Lines are perpendicular to AB; CD is parallel to EF. - - - - Ax. 10. 1. and conſequently, the Planes CGDH and EIFK cannot meet, being produced towards D and E, or F and C.

Th. the Planes CGDH and FIEK are parallel - Def. 2.

THEOREM VII. 15 Euclid.

If two Right Lines, cuting or meeting each other, be parallel to two other Right Lines, alſo meeting each other, not in the ſame Plane with the firſt two; Planes paſſing through each two Lines, reſpectively, are parallel.



Let, the Right Lines AB, and BC, meeting at B, be parallel, reſpectively, to DE and EF, meeting at E, and not in the ſame Plane with AB, and BC.

The Planes ACB, DFE, drawn through thoſe Lines, are parallel.

From the Point B (in which AB meets BC) draw BG perpendicular to the Plane DFE (Pr. 1. 7.) and, through the Point G, in which BG cuts that Plane, draw, GH and GI, reſpectively, parallel to ED and EF. (Prob. 5. 1.)

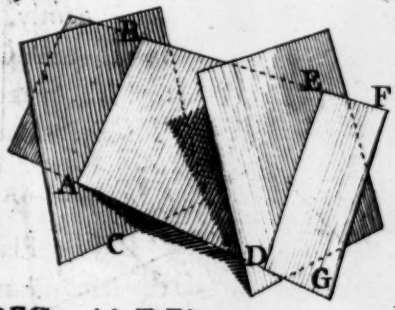
DEM. Then, becauſe BG is perpendicular to the Plane DEF
GH, and GI make Right Angles with BG. - Th. 2.
But, AB is parallel to GH, and BC to GI; - - 5.
(for, they are reſpectively parallel to ED and EF)
And, BGH, BGI are Right Angles; - - Th. 2.
conſequently, ABG and CBG are Right Angles - 4. 1.
wherefore, BG is perpendicular to the Plane ABC; - 2. 7.
and, conſequently, the Plane ACB is parallel to DFE, in which the Lines AB, BC, and DE, EF are ſituated. } 6.

THE-

THEOREM VIII. 16 Euclid.

If two, or more, Planes, which are parallel, be cut by another Plane; their common Sections are parallel.

This Theorem might paſs for an Axiom, or be deduced from the 6th, as a Corollary; but it is eaſily demonſtrated.



Let the Planes ABC and DEG be parallel; and, imagine a Plane, ABFG cuting them, in AB and DE.

I ſay, the Sections, AB and DE, are parallel to one another.

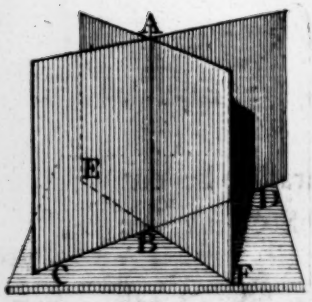
Dem. If AB (being in the ſame Plane, ABFG, with DE) be not parallel to DE, they will meet, if produced, towards one extreme on the other. - - Def. 7. 1.
But, AB is in the Plane ABC; and DE, in DEG - Th. 1.
And, the Plane ABC is parallel to DEG. - - Hyp.
Therefore, ſince both Lines are in the Plane ABFG, and are, alſo, reſpectively, in the Planes ABC, DEG, AB is parallel to DE; ſeeing they cannot go out of thoſe Planes, and conſequently can never meet. - Ax. 1.

THEOREM IX. 18 Euclid.

If a Right Line be perpendicular to a Plane; every Plane, which paſſes through that Line, is alſo perpendicular to the Plane.

Let AB be perpendicular to the Plane CEDF, and let any Plane CAD, or EAF, paſs through the Line AB.

I ſay,



I ſay, the Planes, C A D and E A F, are perpendicular to C E D F.

Dem. Now, becauſe A B is perpendicular to the Plane C E F (by Hypotheſis) it does not incline to it on any Side-Def. 3.

But, the Line AB is in the Plane CAD; and it is alſo in the Plane E A F; (by Suppoſition). conſequently, thoſe Planes do not incline, on either Side, to the Plane CEF; ſeeing that, every Line, BC, BE, BD, and BF, make Right Angles with AB. - - - Th. 2. Th. the Planes, CAD, EAF are perp. to CEF - Def. 4.

Corollary I. The 19th Propoſition of Euclid.

If two Planes, cuting each other, be perpendicular to the ſame Plane; their common Section is perpendicular to that Plane. The Converſe.

For, ſince the Planes CAD, EAF, cuting each other in the Line AB, are perpendicular to the Plane CEF, AB is perpendicular to CEF.

Becauſe, there cannot be drawn from the ſame Point, B, on the ſame ſide of the Plane CEF, two Right Lines, perpendicular to the ſame Plane. And AB is in both Planes - - Th. 1.

Therefore, AB, being in both Planes, CAD, EAF, is perpendicular to the Plane CEF. - - - Cor. to Prob. 2.

Corollary II. The 38th Propoſition of Euclid.

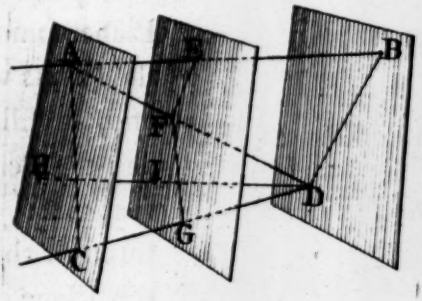
If from any Point in a Plane, which is perpendicular to another Plane, a Right Line be drawn, perpendicular to that other Plane; it will cut the other in the common Section of the two Planes.

For. ſince one Plane is perpendicular to the other (Hyp.) every Line drawn from any Point, in one, perpendicular to the other, is wholly in the firſt Plane; as AB.

Conſequently, ſince it cannot go out of that Plane, (Ax. 1.) it muſt neceſſarily cut the other Plane in their common Section.

THEOREM X. 17 Euclid.

Right Lines, being cut by parallel Planes, are cut proportionally.



Let the Right Lines, AB and CD, be cut by the Planes ACH, EG, and BD (being parallel between themſelves) in the Points A, E, B, and C, G, D.

I ſay, as AE is to EB, ſo is CG to GD.

Join the Points A and C, B and D; and, AD being drawn diagonal-wiſe, cuting the Plane EG in F, draw EF and FG, joining the Points where they cut the Plane EG.

Dem. Then, becauſe AB, AD, and BD are three Right Lines, cuting one another, they are in the ſame Plane ABD; and ADC is alſo a Plane, by the ſame - Ax. 1.

Now, ſince the Planes BD and EG are parallel, and are both cut by the Plane ABD;
their common Sections, BD and EF are parallel;
and, for the ſame reaſon, AC is parallel to FG. } Th. 8.
But, in the Triangle ABD, becauſe EF is parallel to BD, the Sides AB, AD are cut, as AE : EB :: AF : FD - 2. 6.
And, becauſe FG is parallel to AC, as AF : FD :: CG : GD.
Therefore, as AE is to EB, ſo is CG to GD. - Ax. 13. 5.

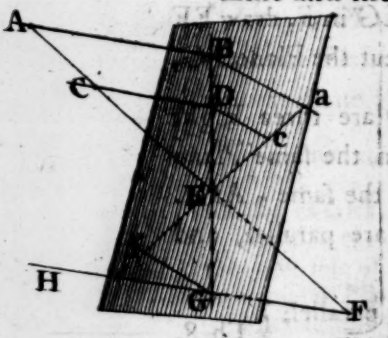
Cor. Hence it is manifeſt, that any Number of Right Lines, proceeding from one Point, are cut proportionally (from their common Point of Section) by parallel Planes.

For, ſince, DF : DA :: DG : DC; 2. 6. & 6. 5.
conſequently, any other Right Line, DH, is cut in the ſame Ratio, at H and I.

THE-

THEOREM XI.

If two Right Lines are perpendicular to a Plane, (whether they are on the ſame Side, or on contrary Sides,) and are both cut by another Right Line, which alſo cuts the Plane; that Line will cut the Plane, ſomewhere in a Right Line paſſing through the Points where the Perpendiculars cut the Plane; and, it will be cut in the ſame Ratio, by the Plane, and the Perpendiculars, as the Perpendiculars, by that Line; and the Right Line, joining or paſſing through the Points where the Perpendiculars cut the Plane, will alſo be cut proportionally, by that Line and the Perpendiculars.



First; let the two Lines AB and CD, (on the ſame Side) be perpendicular to the Plane a B f; cuting it in the Points B and D.

From any Point A, in AB, draw a Right Line AC, cuting CD, in C, and produce it, till it cuts the Plane at E.

Then, a Right Line, BD, being produced, will paſs through the ſame Point, E; and, they will, both, be cut, in the Points A, C, E, and B, D, E, in the Ratio of the two Perpendiculars; viz. as AB to CD.

Dem. For, becauſe AB and CD are both perpendicular to the ſame Plane, they are parallel. - - - Th. 3.
wherefore, AC and BD are in the ſame Plane. - Ax. 3.
And, becauſe the Points B and D are in the Plane a B f; the Right Line BD, is in that Plane - - Ax. 1.
(for it is the common Section of the two Planes)
Conſ. every Line (as AC) not parallel to BD, in the Plane AEB, will cut the other Plane, in the Interſection of both.
Th. AC produced, will cut the Plane a B f, in B D produced.

2nd. Becauſe CD is parallel to AB, the Sides EA, EB, of the Triangle AEB, are cut proportionally. - - 2. 6.
wh. as EC:CA::ED:DB; conſ. EC:EA::ED:EB -6.5.
i. e. EC:EC+CA::ED:ED+DB; viz. as CD:AB -4.6.

Again. Let FG (on the other Side of the Plane) be perpendicular to the ſame Plane (a B f) with AB; they are parallel (3.) which is evident, producing either; as FG, to H

Then, if AF be drawn, cuting AB and FG, it will alſo cut the Plane a B f in the Line BG, joining the two Points, B and G, in which the Perpendiculars cut the Plane.

And, they alſo, mutually, cut each other, in the proportion of AB to FG.

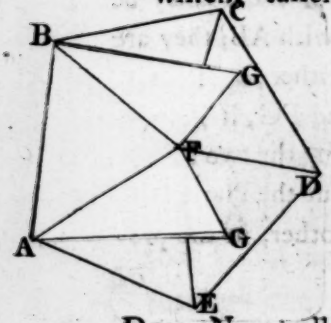
Dem. Becauſe A B is parallel to F G, a Plane may paſs through them both. - - - - - - Ax. 2.
conſequently, AF and BG are both in the ſame Plane with AB and FG. - - - - Ax. 3 and 4.
Wherefore, in the Triangles, EAB, EFG, the Angles at E being vertical, are equal (2. 1.) the Angles at B and G are right (2. 7.) therefore equal - - Ax. 10. 1.
the Angles at A & F, are conſ. equal; and alſo, by Th. 4. 1.
Therefore, thoſe Triangles are ſimilar; - - 4. 6.
and therefore, as AB : FG : : AE : EF, and BE to EG.

Cor. Hence it is manifeſt, that if a Right Line, a B, be drawn, at pleaſure, through the Point B, in the Plane a B f; and through G, if f G be drawn parallel to a B; a B and f G being made equal to AB and FG, reſpectively, or in the Ratio of AB to FG; then, a Right Line, drawn through a and f, will cut BG in the ſame Point E, as before.

For, becauſe a B is parallel to f G, the Triangles E a B, E f G, are ſimilar; or, becauſe a B is parallel to f G, a f and BG are cut proportionally, in E. - - - - - Caſe 3. 2. 6.
And, becauſe the Triangles are ſimilar, a E : E f : : BE : EG, and, as a B : f G. - - - - - - 4. 6.
But, a B : f G : : AB : FG; conſ. a E : E f, and BE : EG, : : AB : FG; and therefore, BG is cut in the ſame Point (E) as before.

THEOREM XII. 21 Euclid.

Every ſolid Angle is contained under Plane Angles; which, taken together, are leſs than four Right Angles.



Let ABCDE be a Section of the Planes AFB, BFC, &c. forming a ſolid Angle, at F; i. e. let ABCDE be the Baſe of a Pyramid, whoſe Vertex is F; ſuppoſed to be elevated out of the Plane of the Baſe.

DEM. Now, all the Angles AFB, BFC, &c. being in a Plane, are equal to four Right Angles. - - Cor. 2. 2. 1.

But, if F be ſuppoſed elevated out of that Plane, equal to FG, and perpendicular to AF, BF, &c. and AG, or BG, drawn, AG is longer than AF - - - 12. 1.

For the ſame reaſon, all the Right Lines, from each Angle, A, B, &c. of the Baſe, to the Vertex, at F, elevated perpendicularly, at F (eq. FG) are longer than BF, CF, &c.

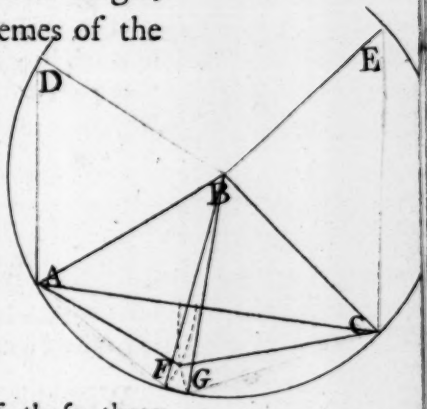
But, if two Sides of a Triangle be, reſpectively, greater than two Sides of another Triangle, and the remaining Sides are equal, they ſhall contain a leſs Angle - 14. 1.

Therefore, the Angles of the Triangles AFB, BFC, &c. being conſidered as raiſed out of the Plane ABCDE, are leſs, reſpectively, than thoſe Angles in the Plane; which, being equal to four Right Angles, conſequently, the Angles elevated out of the Plane, are leſs than four Right Angles.

If the Triangles AGF, BGF, be turned up, on AF & BF, till FG, in one, coincide with FG in the other; and from the Point G, ſo elevated, (FG being perpend. to the Plane ABD) if Right Lines are drawn to all the other Angles, C, D, and E; each two will contain a leſs Angle than thoſe on the Plane, to which they are oppoſite, (as AGB than AFB) and are, together, leſs than four Right Angles.

THEOREM XIII. 20 & 22 Euclid.

If any two, of three given Plane Angles, be greater than the third, and the Lines containing the Angles all equal; and if thoſe three Angles are, together, leſs than four Right Angles, they will form a ſolid Angle; and three Right Lines, joining the extremes of the equal Sides, will form a Triangle.



Let ABC, ABD, and EBC be three Plane Angles; of which, any two are greater than the remaining Angle; and, the Sides, AB, BC, BD, and BE are all equal; alſo, the three Angles, together, are leſs than four Right Angles.

Then, a ſolid Angle may be formed of thoſe three Angles, ABC, &c. alſo, of the three Lines, AC, AD, and EC, a Triangle may be formed.

DEM. Turn over the Triangles ~~AFB and BGC (equal~~ ABD and EBC) till the two Sides BF and BD coincide.

Becauſe they are leſs than four Right Angles (the deficiency being the Angle EBD) it is manifeſt they will form a ſolid Angle, at B, their common Vertex; the two leſſer Angles, ABD, CBE being together greater than ABC.

For, it is evident, if they were leſs, as AB*F*, CB*G*, the Sides B*F* and B*G* would not meet each other; and conſ. with ABC, they cannot form a ſolid Angle, at B.

Therefore, a ſolid Angle formed of three Plane Angles, any two are, together, greater than the third. Q. E. D.

Or, a ſolid Angle formed of any number of Plane Angles, any one muſt be leſs than all the others, added together.

Nothwithſtanding, if B*F* and B*G* be equal to AB & BC reſpectively, although the Angles (AB*F*, CB*G*) are, together, leſs than ABC, A*F* & C*G*, together, are greater than AC; and conſequently, a Triangle may be formed of the three Lines AC, A*F* and C*G*; and therefore, of AC, AD, and EC; which are greater than A*F* and C*G* - 13. 1.

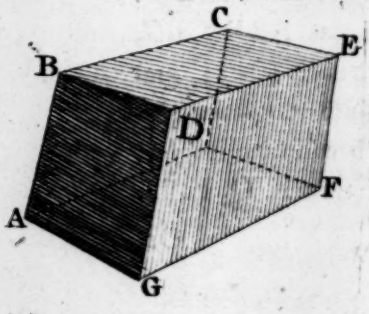
N. B. The laſt part of this Theorem, which (according to Euclid) ſeems to imply, that, unleſs two of the Plane Angles are greater than the third, a Triangle cannot be formed of the three Lines, joining the extremes of equal Sides; whereas, the contrary is manifeſt. It appears however, to me, of ſo little conſequence, as not worthy of being made a diſtinct Theorem, of itſelf.

N. B. 2. To make a Solid Angle of three given Plane Angles, (the 23rd Prop. of Euclid) is to place them ſo, together, that the Vertices, of all the three, meet in one Point; and, the Sides, which contain the plane angles, coincide, two and two in one Line, as in the Figure. (ſee Defin. 6.)

With this Problem, Euclid's Commentators take up more than four Pages; but, for what purpoſe I can't deviſe.

THEOREM XIV. 24 Euclid.

If a Solid be contained within ſix Planes, of which two and two are parallel; the oppoſite Planes are Parallelograms, equal and ſimilar.



Let ABEG be a Parallelopiped.

The oppoſite Planes ABCD, DEFG; BCED, and ADFG; alſo, ABDG, and DCEF, are, reſpectively, equal and ſimilar.

DEM.

DEM. The Planes, AC, GE, being parallel, and both cut by the Planes BG, and CF, which are alſo parallel; their Sections, AB and DG, CD and EF, are parallel amongſt themſelves. - - - - - - - Th. 8.

But, they are alſo cut by the parallel Planes, BE and AF; their Sections, AD, BC, DE, and FG, are conſ. parallel.

Therefore, the Figures formed by thoſe Sections are Parallelogms. ſeeing their oppoſite Sides are parallel-Def. 33

2nd. Becauſe AB is parallel to GD, and AD to GF, and are in parallel Planes, the Angle BAD is equal to DGF; wh. the oppoſite Angles, BCD and DEF, are equal-15.1. But AB = CD, and alſo to DG; and DG = EF-ſame. conſ. they are all equal amongſt themſelves. - Ax. 3. 1. Alſo, AD, BC, DE, and GF are equal amongſt themſelves; therefore, the Parallelogram ABCD = DEFG, and ſimilar; alſo, BE = AF, and BG to CF, and reſpectively ſimilar.

THEOREM XV. 25 Euclid.

If a Parallelopiped be cut in two Parts, by a Plane, parallel to any two of its Planes; the two Solids will have that Proportion to each other, as their Baſes (on which they are ſuppoſed to ſtand) i. e. as the Parts of any Plane cut by the other; or, as the Segments of the Sides.

This Theorem is manifeſtly an Axiom.

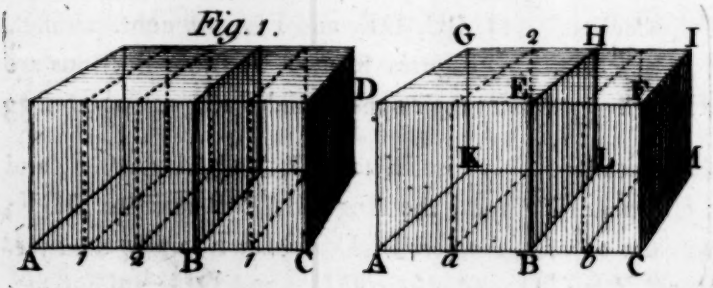
For, ſince the Sections made by parallel Planes are equal, every where, which is manifeſt, ſeeing the oppoſite Planes are equal and ſimilar; conſequently, if each Part be again cut, into Parts, by parallel Sections, which are equal amongſt themſelves (as in Fig. 1.) the Demonſtration follows from the 9th Axiom of the 5th Book.

Or,

Or, being incommenſurable, they may be divided into an equal Number of Parts, (Fig. 2.) and demonſtrated as Theorem 1ſt. 6.

If the Baſe (AKMC) or Sides (AC, or DF) are biſected, the Parallelopiped is biſected; and conſequently, in whatever Ratio one is divided, (by a Plane BEHL) the other is the ſame.

Fig. 1.



DEM. Let the Baſe AM, or any other Plane, AF, be cut by a Right Line, parallel to its Sides, AK or AD; and imagine a Plane (BH) to paſs through BE or BK, parallel to AG; then, AE : EC, or AL : LC :: AB : BC. - - Th. 1. 6. And, ſeeing the oppoſite Planes, DGIF and GM, are equal and ſimilar, to AKMC, and AF, reſpectively; and are all cut by the Plane BEHL, parallel to AG and CI; conſequently, the Solid AGHB : BHIC :: AE : BF, or, AL : BM, i. e. as AB : BC.

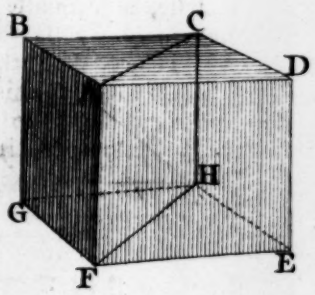
N. B. The two following Propoſitions, 26th and 27th of Euclid, are Problems; which, I do not conceive to be at all neceſſary, or elementary.

To make Solids, is rather a mechanical Proceſs than geometrical; 'tis ſufficient, for the purpoſe of Demonſtration, to ſuppoſe that one ſolid Angle, or one Solid is equal to another; for, an attempt to make them really ſo, on a Plane, is inconſiſtent; unleſs it be to delineate them, perſpectively ſo.

THE-

THEOREM XVI. 28 Euclid.

If a Parallelopiped be cut by a Plane, paſſing through the Diagonals of two oppoſite Planes, it will be cut in two equal Parts.



Let ABDF be a Parallelopiped, and AC, FH, Diagonals of the oppoſite Planes, ABCD and EFGH.

Then, becauſe AF and CH are both parallel to BG, they are parallel between themſelves (Th. 4.) wherefore, AF and CH are in the ſame Plane - Ax. 3.

I ſay, the Parallelopiped, ABDF, is biſected by the Plane ACHF.

Dem. For, the Triangle ABC=ADC; FGH=FEH - 15. 1.
And the Parallelogram ABCD is equal to EFGH - Th. 14. (wh. the Triangles are all equal amongſt themſelves.)
But, the Parallelogram BF=CE, & AE=BH; - Th. 14.
alſo, the Triangles ABC, and FGH; ADC, and FEH are equal to one another - - - - above.
wherefore, the Priſm FBC is equal to CEF - Ax. 5.
(For, they are contained within the ſame number of Planes which are equal, and ſimilar to each other, reſpectively)
Therefore, the Plane, ACHF, paſſing through oppoſite Diagonals, (AC and FH) biſects the Parallelopiped, ABDF.

Otherwiſe.

By the parallel motion of the Parallelogram ABCD along the Right Lines AF, DE, &c. the Parallelopiped is ſuppoſed to be generated. (ſee Note, Def. 8.)

Wherefore, ſince the Triangles, ABC, ACD, are equal, and the Priſm, FBC, CEF, are deſcribed in equal Time, that is, by the ſame motion of both together, conſequently, they are equal.

The Definition of a Priſm, as it is given by moſt Authors, is ſomewhat obſcure and unſatisfactory; and, we are left to imagine, that any one of the Planes may be its Baſe; we may as reaſonably conclude, that any Plane of a Pyramid may be its Baſe, but it is never ſuppoſed ſo, unleſs they are all Triangles; nor in a Priſm, except they are all Parallelograms.

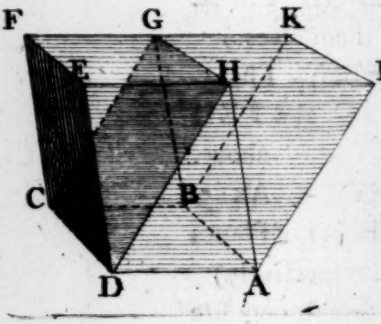
For, to ſuppoſe the two equal, ſimilar, and parallel Planes, in a Priſm, Triangles or Poligons, and any other Plane to be its Baſe, is inconſiſtent.

Hence, the 40th Propoſition of Euclid, which, in reality, is but a Corollary to this Theorem, is an abſurd Propoſition.

Becauſe, I do not conceive, that, unleſs a Priſm be alſo a Parallelopiped, its Baſe can be a Parallelogram.

THEOREM XVII. 29, 30, & 31 Euclid.

Parallelopipeds having the ſame Baſe,* or equal Baſes, and the ſame Altitude, are equal to one another.



First; let the two Parallelopipeds, AHFC and AIGC, have the ſame Baſe, ABCD; and, let their oppoſite Faces have a common Side, GH.

I ſay, the Parallelopiped AHFC is equal to AKGC.

---

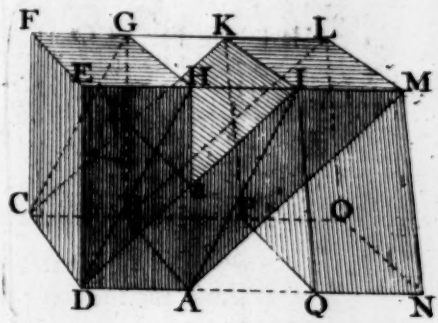
* By the Baſe, of a Solid, literally, is underſtood the Plane or Face, on which it is ſuppoſed to reſt; but, it is not always conſidered ſo, in Geometry,

When any two Faces of Parallelopipeds, are in one Plane, or in parallel Planes; whether they coincide with each other, entirely, or are apart, and diſtinct from each other (no regard being had to the poſition of the Plane they are in) the Top, in ſuch Caſe, is ſometimes called the Baſe.

N. B. Parallelopipeds, having the ſame Altitude, are ſuppoſed to be contained between the ſame parallel Planes; i. e. two of their oppoſite Planes, in each Solid, are in the ſame Plane; or, that there is the ſame perpendicular diſtance between them.

Dem. The Parallelopiped AHFC, being cut by a Plane, CGHD, through the oppoſite Diagonals, DH and CG, is cut into two equal Parts CAG = DFH. - Th.16.
And, for the ſame reaſon, the Plane AHGB biſects the Parallelopiped AIGC; in AIKG equal ADCG.
But, ADCG + DFH = ADCG + AIKG - Ax.6.1.
Therefore, the Parallelopiped AHFC = AIGC. Q.E.D.

Secondly. Let the Parallelopipeds, AHFC and AMKC, have the ſame Baſe, ABCD; their oppoſite Faces, EFGH and IKLM, are in the ſame Plane, and have not a common Side.
AMKC is equal to AHFC.

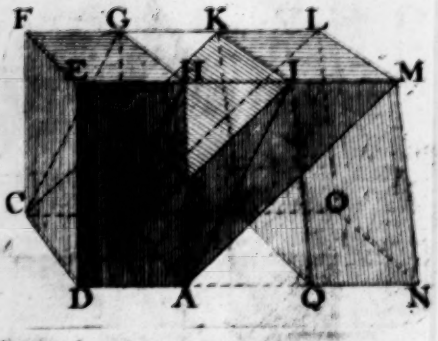


Let the ſpace, GHIK, between the upper Faces, be equal to the Baſe ABCD; conſ. equal to EFGH, equal IKLM-Th.14.

Dem. The Parallelopiped AHFC is equal to AIGC; -proved.
and, for the ſame reaſon, AIGC is equal to AMKC.
Th. the Parallelopiped AMKC is equal to AHFC-Ax.3.1.

Again. Suppoſe the ſpace, GHIK, be either greater or leſs than EFGH.
The Triangles, AMH, DIE, are congruous; - 7. 1.
(for AM is equal to DI, MH = IE, and AH = DE; by 15.1.)
And the Sides, AB, DC, EF, GH, IK, & LM, are equal -14.
the Par. AG = DF; ABLM = IKCD; & EFKI = MLGH;
conſ. the Solid AGLM is equal to DFKI. - Ax. 5.
Now, if the Solid, aIKGb (common to both) be taken away,
AMKb is equal to DaGC. - - Ax.7.1.
And, if the common Solid AabCD be added, to both;
the Parallelopiped, AMKC, is equal to AHFC - Ax.6.1.

The ſecond Part, viz. when they have equal Baſes only, and have not one common to both, is readily proved.



Let the Baſe, ABCD, of the Parallelopiped AHFC, be equal to Baſe NOPQ, of the Parallelopiped NMKP (and ſimilar)

Then, the Parallelopiped, AHFC, is equal to NMKP.

Let them be ſo ſituated, that the Planes AE, and NI, are in the ſame Plane, DEMN; and draw AM, DI, BL, and CK.

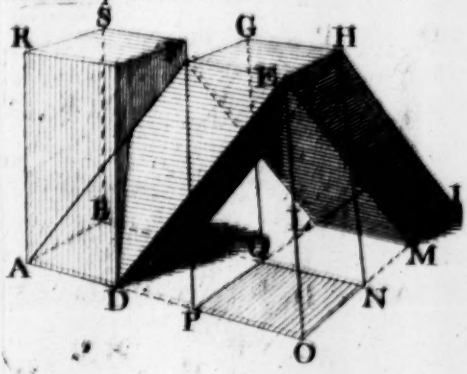
DEM. Now, becauſe MI is parallel to AD, and LK to BC, and alſo equal, AMID and BLKC are Parallelgms - C. 15. 1. And, for the ſame reaſon, AMLB and DIKC are Parallelograms; and they are, alſo, parallel and ſimilar, reſpectively; wherefore, AMKC is a Parallelopiped. - - Def. 9. (For, AM is parallel to DI, and ML to IK; wherefore, the Angle AML = DIK, &c. - Th. 5.)

But, the Parallelopiped AHFC has the ſame Baſe with AMKC; and they are contained within parallel Planes; wherefore, they are equal Parallelopipeds. - - proved. Alſo, the Parallelopiped NMKP, has the ſame Baſe with AMKC (i. e. the Plane or Face IKLM is common to both.) conſequently, they are equal Parallelopipeds. - ſame. But, the Parallelopiped AMKC is, alſo, equal to AHFC. Therefore, the Par. NMKP, is equal to AHFC - Ax. 3. 1.

After the Proofs already given, it may, by ſome, be thought unneceſſary to add more; but, there are other Caſes, in which there is abundantly more room to diſpute the Aſſertion, than in the preceding; and, as this Theorem is very eſſential, it cannot be too much enforced.

CASE II.

CASE II. When no more than two Faces of the Parallelopipeds (i. e. two in each) are in the ſame parallel Planes.



Let AGHD and FHIL be Parallelopipeds; whoſe Baſes, ABCD and IKLM, are equal, and in the ſame Plane; and the Top, EFGH, common to both; but have no other Plane common.

I ſay, that the Parallelopiped AGHD, is equal to FHIL.

Produce AD and BC, IM and KL, (being in the ſame Plane) they will cut each other, in NOPQ, which is a Parallelogram; equal and ſimilar to ABCD, and IKLM.

DEM. Becauſe EFGH is common to both Parallelopipeds, and becauſe the oppoſite Planes are equal and ſimilar, AC is ſimilar to LI; and, being ſimilarly poſited, IK, IM, are parallel to AD and BC; and IM, KL, to AB and DC; therefore, NOPQ, being between the ſame Parallels, is conſequently equal and ſimilar to AC, & LI - 4. & 18. 1. and conſequently, to EFGH; and alſo ſimilarly ſituated.

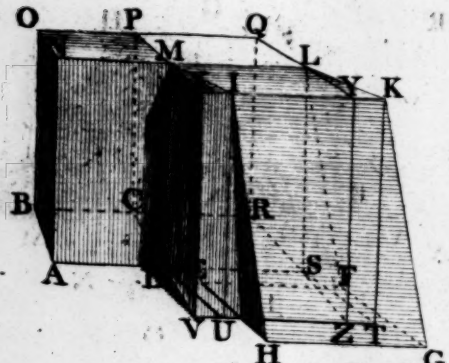
Join EO, FP, GQ and HN.

Then, PFHN is a Parallelopiped; which, being equal to each of the other, AGHD and FHIL (part 1ſt) they are conſequently equal between themſelves - Ax. 3. 1.

Alſo; if the Parallelopipeds, ASFD and FHIL, have their Baſes equal; and have no Face common; being between the ſame parallel Planes, AOI and RGE, they are equal Parallelopipeds.

This is evident, from the foregoing; each being equal to PFHN.

CASE III. When the Parallelopipeds have no Face common, nor ſimilar Figures.



Let the Parallelopipeds AOMD and EMKG, be between the ſame parallel Planes, ARH and NQI; conſequently, they have equal Altitude; and, let the Baſe, ABCD, be equal to the Baſe, EFGH, of the other, which are not ſimilar Figures.

I ſay, the Parallelopiped, EMKG, is equal to AOMD.

DEM. Let the Side NM, of one, and ML of the other, be ſo placed, at the common Angle M, that NML is a Right Line.

Compleat the Parallelopiped, DMKT; having a common Side, DM, with AOMD.

The Parallelopiped DMKT is equal to EMKG-Caſe 2nd. (For they have the ſame Altitude, and IMLK is common)

Produce KI and TU, PM & CD, meeting at X and V. Draw LY and SZ, parallel to MX and DV, cuting IK and TU, at Y and Z. Join VX and YZ.

Then the Parallelopiped VMYZ = UMKT - 1ſt. conſ. the Parallelopiped, VMYZ = HMKG - Ax. 3. 1.

Produce OP and YL, BC and ZS, meeting at Q and R; and, joining QR, there is formed a Parallelopiped DPQS; equiangular to both the Parallelopipeds, AOMD and VMYZ; conſ. they are equiangular amongſt themſelves.

Now, the Parallelogram IMLK = NOPM - Hyp. XMLY (equal IMLK 18. 1.) is equal to NOPM - Ax. 3 1. conſequently, (the Angle PMN being equal XML - 2. 1.) as NM : ML :: XM : MP. - - - 8. 6.

But, the Parallelopiped AOMD : DPLS :: NM : ML
And, the Parallelopiped VMYZ : DPLS :: XM : MP; } 15.
i. e. as NM : ML. - - - - - above.
Wherefore, the two Parallelopipeds, A O M D and V M Y Z, have an equal Ratio to D P L S;
and therefore they are equal. - - - Ax. 4. 5.
But, the Parallelopiped H M K G = V M Y Z - above.
Therefore, H M K G, is equal to A O M D.

COROLLARY I. The 33rd Propoſition of Euclid.

Parallelopipeds having equal Altitudes, have that Ratio to each other as their Baſes.

For, ſince by this Theorem, all Parallelopipeds, having equal Baſes and equal Altitudes are equal; and, by the 15th, if a Parallelopiped be cut in two, by a Plane, parallel to its Planes, the two Parts have that proportion to each other as their Baſes; conſequently, all Parallelopipeds, having equal Altitudes, are to each other as their Baſes.

COR. II. Parallelopipeds, having equal Baſes, have that Ratio to each other as their Altitudes.

For, having equal Baſes and equal Altitudes, they are equal; conſequently, if the Altitude of one, be half, or a third part, &c. of another; or, in whatever Ratio the Altitude of one Parallelopiped is to the other, whoſe Baſes are equal, they are in that Proportion, to each other, as their Altitudes.

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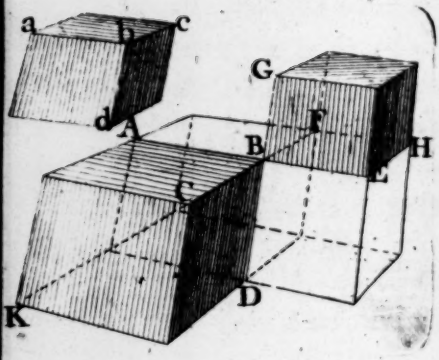
THEOREM XVIII. 33 Euclid.

The Proportion of ſimilar Parallelopipeds, to one another, is the triplicate Ratio of their correſponding Sides.



Let ADK and a c d be ſimilar Parallelopipeds; in which, let AB, a b, BC, b c, and BD, b d be correſponding Sides, in each Solid.

Then the Proportion, which the Solid ADK has to a c d, is the triplicate Ratio of AB to a b, BC to b c, or BD to b d.

In the Solid ADK, produce the Sides, AB, CB, & DB. Make BE equal to a b, BF equal b c, and BG equal b d; draw EH and FH, reſpectively, parallel to BF and BE; and compleat the Parallelopipeds GH, DH, and AFD; equiangular to A D K.

DEM. Becauſe B H is a Parallelogram (by Conſtruction) and the Angle EBF = ABC, (2.1.) (EBG = ABD, and FBG = CBD) alſo BE = a b, and BF to b c; BH is equal and alſo ſimilar to a c; for, a c is ſimilar to AC; by Hyp.

And, for the ſame reaſon, GE, is equal and ſimilar to a d (which is ſimilar to AD) alſo, FG is equal and ſimilar to c d (ſimilar to CD) conſequently, the Parallelopiped, GH, is equal and ſimilar to a c d (ſimilar to AKD)

Now, AB:a b::BC:b c, and as BD is to b d-Def. 11.7. and, the Parallelopiped G H is equal and ſimilar to a c d; BE = ab, &c. wh. as AB:BE::CB:BF, and DB:BG. But, the Parallelopipeds,, A F D and D F H, have equal Altitude, with A D K.

wh. they have that Proportion to each other as their Baſes;
i. e. the Solid ADK : AFD : : Par. AC : AF; i.e. as CB to BF } Cor. 1.
and, the Solid AFD : DFH : : Par. AF : FE; i.e. as AB to BE } of 17.
alſo, - - DFH : GEH : : Par. DE : EG; i.e. as DB to BG } & 1.6.

wherefore, the Parallelopiped ADK : AFD : DFH : GEH ÷ in continual Ratio. (for the Ratio of CB to BF, AB to BE, and of DB to BG, is the ſame, by Hypotheſis.)
Therefore, the Ratio of ADK to a c d (equal GH) is triplicate of ADK to AFD; i. e. of CB to BF (eq. b c) of AB to BE (eq. a b) or, of DB to BG (eq. b d)-Def. 12. 5.

THEOREM XIX. 34 Euclid.

The Baſes and Altitudes, of equal Parallelopipeds, are reciprocally proportional.

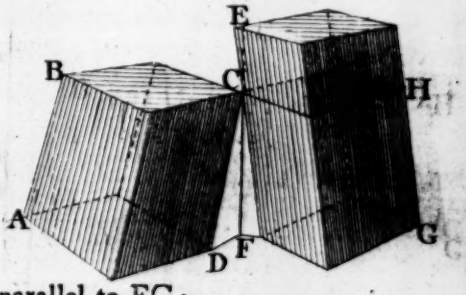
This Theorem will admit of as many Caſes as the 17th; but one general Caſe, being clearly demonſtrated, will be ſufficient.

Let ABD and FEH be equal Parallelopipeds.

If the Baſe of one be equal to the Baſe of the other, their Altitudes are alſo equal (Th. 17.) becauſe the Parallelopipeds are equal, by Hypotheſis. Let them be unequal.

Then as the Baſe AD is to FG, ſo is E F to C F, their Altitudes.

Let them be ſo placed together, that the Angle C, of the one, touches the Side E F, of the other.

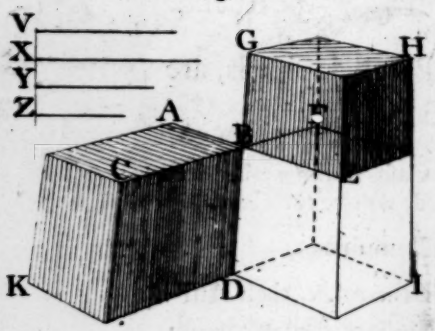


Produce the Plane, BC, to H, parallel to FG; and, through C, let E F be drawn, perpendicular to their Baſes, AD, FG (which are ſuppoſed to be in the ſame Plane) EF and CF are the Altitudes of the Parallelopipeds.

Dem. Then, ABD : FCG : : AD : FG, their Baſes - Cor. 1. 17
And, - FEG : FCG : : EF : CF, their Altitudes - Cor. 2. 17
But, - ABD is equal to FEG; by Hypotheſis.
Therefore, - ABD : FCG, i. e. AD : FG : : EF : CF
i. e. as the Baſe AD, of one, is to FG of the other;
ſo is EF the Altitude, of the laſt, to CF of the firſt.

THEOREM XX.

Parallelopipeds, whoſe ſolid Angles are equal, have that Proportion, to one another, which is compounded of the Ratio of their Sides.



Let ADK and BH be equiangular Parallelopipeds; and let them be ſo placed together, at the equal Angles B, that, the Side AB of the one, is in the ſame Right Line with BE, of the other, and CB with BF; conſequently, the two Sides BD and BG are in one Right Line. (It is not material which Plane, EF, GE, or GF, is made the Baſe.)

On EF, and BD, compleat the Parallelopiped DFI.

Take any Right Line, V, at pleaſure; and make X to, V as BE to AB; and as BF is to CB, ſo make Y to X; alſo, as BG is to BD; make Z to Y. - (Prob. 32)

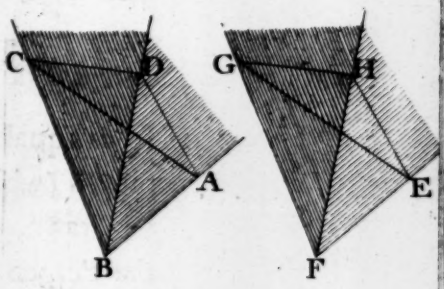
Then, as V is to Z, ſo is the Parallelopiped, ADK to BH.

DEM. The Solids, ADK and DFI, have equal Altitudes; wherefore, they are to each other as AC to EF; -C.1.17.
that is, as the Ratio which is compounded of AB to BE, and CB to BF; i. e as V to Y. - - Th.11.6.
And, the Solid DFI is to BH, as the Par. DE is to EG - 17.
that is, as DB is to BG, i. e. as Y to Z. - 11. 6.
But, ADK: DFI :: V: Y, and DFI: BH :: Y: Z -Con.
Therefore, the Solid ADK: BH :: V: Z - - Th. 9. 5.
i. e. in the compounded Ratio of AB to BE, CB to BF, and DB to BG.

THE O-

THEOREM XXI.

In ſolid Angles contained by three Plane Angles, two and two of which (i. e. one in each) being equal, reſpectively; thoſe Planes, which contain the equal Angles, have equal Inclination to one another.



Let B and F be two ſolid Angles, each contained by three Plane Angles; viz. ABC equal to EFG, ABD equal to EFH, and CBD equal to GFH.

I ſay, the inclination of ABD, or CBD, to ABC, is equal to the inclination of EFH, or GFH, to EFG; alſo, the inclination of ABD to CBD is the ſame, as EFH to GFH.

In AB take any Point, A, at pleaſure, and take EF equal AB; and, in the Planes ABC, ABD; EFG, EFH, draw AC and AD, EG and EH, at right Angles with AB and EF, reſpectively, and join the Points, C and D, G and H, by Right Lines.

DEM. The Angle ABC = EFG (Hyp.) and BAC = FEG - Con. and AB = EF; wh. the Triangles ACB, EGF are congruous; AC is equal to EG, and BC is equal to FG. - Th. 11. 1.
Alſo, the Angle ABD = EFH (Hyp.) & BAD = FEH - Con. wherefore (ſeeing AB = EF) BD = FH, and AD to EH - ſame
And, becauſe BC, BD, are reſpectively equal to FG, FH, and the Angle CBD is equal to GFH; CD = GH - 8. 1
Laſtly, becauſe the Triangle ACD is equilateral to EGH. they are alſo equiangular; and, the Angle CAD = GEH - 7. 1
But, thoſe Angles are the Inclination of the Planes, ABD to ABC, and EFH to EFG. - - Def. 5.
Therefore they have equal Inclination. Q. E. D.

After the ſame manner, it may be proved that any other two have equal Inclination; therefore, &c.

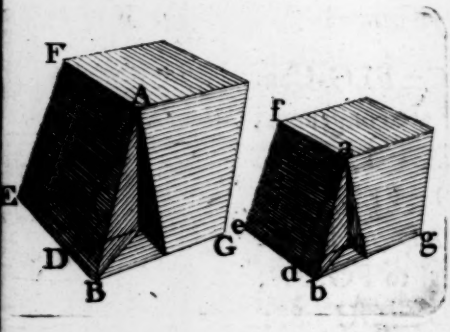
Cor. Solid Angles, contained by three Plane Angles, each, of which, two and two (one in each) are equal, reſpectively, are equal to one another.

Becauſe the Planes have equal Inclination to each other, reſpectively (by the Theorem) Therefore the Angles are equal. *

THEOREM XXII.

If from equal ſolid Angles, in equiangular Parallelopipeds (which are not right angled) Perpendiculars be drawn, ſimilarly, i. e. to equiangular Planes; the Perpendiculars (i. e. their Altitudes) will be proportional to the Sides, which are inclined to the Planes on which the Perpendiculars fall.

This Theorem, though very different, apparently, is the ſame, in Subſtance, as the 35th of Euclid.



Let B F G, b f g be equiangular Parallelopipeds, and let the Angle A be equal to a; conſequently, B is equal to b.

From the equal Angles A, a, let the Perpendiculars AC, a c, be drawn to equiangular Planes (their Baſes) containing the equal Angles E B G, e b g; cuting them at C and c.

I ſay, that, as AB is to a b, ſo is AC to a c.

Through C and c, draw CD and c d, perpendicular to BE and b e, reſpectively; and join AD and ad.

* Profeſſor Simſon ſays (Prop. B.) "and alike ſituated" making that a Condition of their equality; whereas, it is not; for, it is impoſſible, that three Plane Angles, equal reſpectively to three other Plane Angles (containing a ſolid Angle) can be ſo placed, in forming a ſolid Angle, which will not be equal to the other.

For, they muſt neceſſarily have the ſame Inclination each to the other, reſpectively, however ſituated.

Dem. Becauſe AC is perpendicular to the Plane EG - Con. the Plane, CAD, paſſing through AC, is perp. to EG-Th. 9 and, for the ſame reaſon, c a d is perpendicular to e g. But, CD is perpendicular to BE; and c d to b e, - Con. wh. BE is perpendicular to the Plane CAD, and b e to cad; conſ. AD is perpendicular to BE, and a d to b e. - Th. 2.

Then, in the Triangles BAD, b a d, the Angles ADB, a d b are right, therefore equal - - - Ax. 10. 1. and the Angle ABD = a b d, (Hyp) conſ BAD = b a d - 10 1 and therefore, as AB : AD :: a b : a d - - Th. 4. 6.

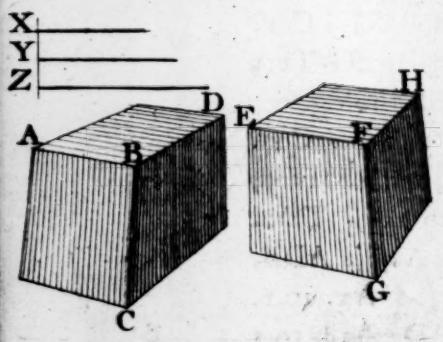
But, the Angle ACD = a c d (becauſe, AC is perpendicular to the Plane EG, and a c to e g, ACD, a c d are Right Angles) and, the Angle ADC = a d c, - 21 for, ADC is the Inclination of the Plane AE to EG; and, a d c is the Inclination of the Plane a e to e g.-Def. 5. wherefore, the Triangles DAC, d a c are ſimilar; and, therefore as AC : AD :: a c : a d. - - Th. 4. 6. But, AD : AB :: ad : ab (above) Th. AC : AB :: ac : ab - 9. 5. and, by alternation, AC : a c :: AB : a b. Q. E. D.

THEOREM XXIII. 36 Euclid.

If three Right Lines are Proportionals; a Parallelopiped deſcribed, or conſtructed under all three, for its Sides, is equal to an equiangular Parallelopiped, whoſe Sides are each equal to the Mean.

If the Parallelopipeds be right angled, that which is conſtructed under the Mean is a Cube.

Let the three Right Lines X, Y, and Z, be Proportionals; as X is to Y, ſo let Y be to Z.



And, let ACD be a Parallelopiped, whoſe Side AB is equal to X, BC equal to Y, and BD equal to Z.*

Alſo, let EGH be a Parallelopiped equiangular to ACD.

Then, if the Sides EF, FG, and FH, are each equal to Y; *
the Parallelopiped ACD is equal to EGH.

DEM. Becauſe, X:Y::Y:Z, the Rectangle under X and Z, is equal to the Square of Y, i.e. $X \times Z = Y \square$ - Cor. 9.6.
But, Parallelograms, having equal Angles, are in the ſame Ratio as Rectangles, whoſe Sides are equal to the Sides of the Parallelograms, reſpectively, - - C. 2. 11. 6.
wherefore, the Parallelogram AD is equal to EH.
But, BC = FG; and conſ. their Altitudes are equal - 22
Therefore, the Parallelopiped ACD is equal to EGH.
For they have equal Baſes, and equal Altitudes.

---

* In thoſe Caſes, the Figures being but repreſentations of Solids, on a Plane; if the Lines, BD, and FH, were made equal to Z, and Y, reſpectively, they could not poſſibly have that appearance, but they would appear much greater; becauſe they are ſuppoſed to recede, from the Planes of the Faces; AC and EG.
Therefore, they are ſuppoſed to be equal, only; which is the ſame thing, and holds equally true, in the Demonſtration. If one be, the other is a neceſſary Conſequence.

Note. As the Ratio, between two ſimilar Plane Figures, is diſcovered by a third Proportional, ſo between Solids, a fourth determines it.

Alſo, to deſcribe plane Figures, in any given Ratio, a Mean Proportional is found; ſo in reſpect of Solids, two Means are requiſite (ſee Prob. 33. or P. 332. B. 6.)

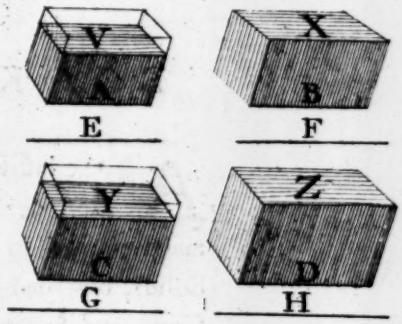
For, ſince the Ratio between ſimilar Solids is triplicate of their correſponding Sides; conſequently, ſince AB : DH : BF : AD ∴; and, if the Ratio of two ſimilar Solids be as AB to AD, AB and DH are in the Ratio of their correſponding Sides.

Hence, ſimilar Solids may be conſtructed in any Ratio to one another.

THEO-

THEOREM XXIV. 37 Euclid.

If four Right Lines are Proportionals; ſimilar Parallelopipeds, and ſimilarly conſtructed on each Line, are alſo Proportionals.



Let A, B, C, and D be Proportionals; as A is to B, ſo let C be to D.

And let the Parallelopipeds V, X, Y, and Z, be ſimilarly conſtructed, on A, B, C, and D.

I ſay, that, as V is to X, ſo is Y to Z.

DEM. Let E and F be ſo taken, as A:B:E:F ∷ } (Prob. 31)
And, let G and H be taken, as C:D:G:H ∷ }
Then, the Parallelopiped V : X :: A : F }
And the Parallelopiped Y : Z :: C : H } - Th. 18.
But, as A:B::C:D; wherefore, as B:E::D:G; and, as E:F::G:H; (Ax. 13. 5) conſ. as A:F::C:H - Th. 9. 5.
Therefore, as the Parallelopiped V:X::Y:Z. Q. E. D.

It is not neceſſary that the Parallelopipeds Y and Z ſhould be ſimilar to V and X; for, if V be ſimilar to X, and Y to Z, the Demonſtration is evidently the ſame.

Or, if V be not ſimilar to X, nor Y to Z;

but, let V be ſimilar to Y, and X to Z; it ſtill holds true.

For, ſince A:B::C:D, ſo A:C::B:D - - Th. 4. 5.

conſequently, ſimilar Parallelopipeds, conſtructed on A and C, and others, not ſimilar to them, on B and D, will ſtill be Proportionals.

For, ſince the Parallelopipeds, on A and C, B and D, are, as V:X::Y:Z, ſimilar; ſo V:Y::X:Z, not ſimilar.

The ſame holds true, in ſimilar Solids of any kind.

ELE-

ELEMENTS OF GEOMETRY.

BOOK VIII. The XII. of Euclid.

AS the laſt Book, or Section, treats only of Plane Solids, and determines their Proportions, their Properties and Relation to each other; ſo, in this, the Doctrine of Solids, bounded by regular curved Surfaces, as the Cylinder, Cone, and Sphere, are conſidered; the Solution of which is truely admirable.

It muſt appear, to all who conſider it, before they have gone through this Book (or ſome other on the Subject), a matter not merely of difficulty, but of abſolute impoſſibility, to aſcertain the true Meaſure or Area of the Surface, much leſs of the ſolid Contents of a Sphere; to reduce ſuch a Solid to right angled and cubical Meaſure, with any degree of certainty, ſeems to be beyond the power of Art, or the reach of finite knowledge.

And yet, nothing more is requiſite thereto, than the true meaſure of its Diameter, which may be acquired; at leaſt of any Sphere formed or conſtructed by Man. In reſpect of the Earth itſelf, its Diameter can only be obtained by its Circumference. It is poſſible, by various means, to come ſomewhat near the truth, but it is beyond human abilities to aſcertain it, with any degree of abſolute certainty.

How much leſs, then, are we able to come at the true Diſtances, by which, the magnitudes of diſtant Planets are reduced to the ſame Scale of Proportion; viz. by the known meaſure

meaſure of the Earth. For, if the one cannot be had, with certainty, how much more incorrect muſt be the other, having no other Data to build on?

Although the many ingenious contrivances, by Inſtruments curiouſly conſtructed for that purpoſe, are worthy of the greateſt praiſe, and redound highly to the Fame of their ſeveral Inventors; and notwithſtanding all other means, by Tranſits, &c. may ſtill approch nearer to the truth, or, at leaſt, evince and prove the imperfections of all human Inventions, for the purpoſe; we muſt (or may as well) remain ſatisfied, that we have acquired ſo much, as to anſwer various great and notable purpoſes, in Trigonometry, Navigation, &c; but, that we ſhall never be able, by any means, to come at the Truth is moſt certain.

The Affinity between the Pyramid and the Priſm, and the Proportion one bears to the other, are, in the firſt place, determined; by means of which, the Proportion between the Cone and Cylinder, is alſo determined.

Laſtly, of the Sphere; ſhewing the Proportion it alſo has to a Cylinder and Cone; as given by Archimedes; for, on that Head, Euclid is totally ſilent; the 12th Book only determines the Ratio one Sphere has to another, in the laſt Theorem. To obtain which, he has two Problems, 17th and 18th; the latter, the moſt prolix of the whole; containing, in four or five full Pages, the moſt tedious deſcription of the Conſtruction of a Solid, which is not of any other uſe, whatever; and previous to it, Profeſſor Simſon alſo gives a preparatory Lemma.

The manner, in which the Ratio of one Sphere to another is here inveſtigated, and deduced from the 9th, is, at once, the moſt ſolid and convincing; and alſo ſhews how the quantity of every Sphere is aſcertained, which the other does not; and therefore, what it does is of little uſe.

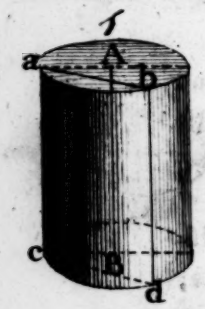
DEFI-

DEFINITIONS.

For the Definition of a Pyramid, ſee Def. 7th of the 7th; which, notwithſtanding it is not uſed there, is the firſt of Plane Solids (having the leaſt number of Faces) and as a Priſm (Def. 8th) includes all Parallelopipeds whatever, it could not be diſpenſed with, though but of little uſe in that Book.

It remains, only, to define ſuch Solids as are bounded by curved Surfaces.

DEF. I. A CYLINDER is a SOLID, having two parallel Plane Surfaces, which are equal Circles; and an uniform convex Surface, bounded, in length, by the circular Planes, and returning into itſelf in width, without bounds. As AB, or *AB*.



DEF. II. The BASES, of a CYLINDER, are the two circular Planes; A and B.

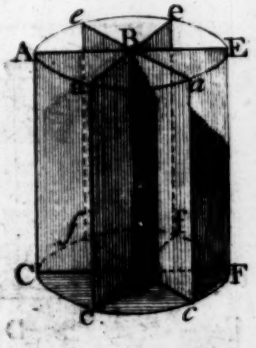
DEF. III. The AXE, of a CYLINDER, is a Right Line paſſing through the Centers of its Baſes. As AB, or *AB*.

N. B. A Cylinder may be conceived to be generated by the Direct motion of a Circle (as A, or *A*) along its Axe, always parallel to itſelf.

Let the Circle A, or *A*, be ſuppoſed to be applied, at its Center, to the Right Line AB, or *AB*; and being moved, parallel to its firſt poſition, at A or *A*, to B or *B*; the Line AB or *AB*, being always in its Center; it will, in that motion, have deſcribed the Cylinder AB or *AB*.

If any Right Line as a b, or *a b*, be drawn in the Circle, cuting the Circumference, at a, or b, *a* or *b*; and, whilſt the Circle deſcribed the Cylinder (the Right Line a b, or *a b*, being always parallel to its firſt poſition) the Point a or b, *a* or *b*, will deſcribe a Right Line, a c, or b d, which is a Side of the Cylinder.

DEF. IV. A RIGHT CYLINDER is one, whoſe Axe, AB, is perpendicular to the Planes of its Baſes, A and B.

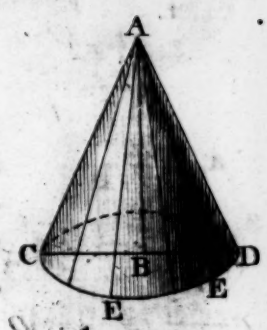


A Right Cylinder may be conceived to be generated by the revolution of a right angled Parallelogram, or Rectangle (CABD) on one Side (BD) which remains fixed; and is therefore called its Axe, or Axis; on which the Rectangle revolves, and on which the Solid, revolving, would appear at reſt.

The two Sides, AB and CD, being equal, deſcribe the equal Baſes, whilſt AC deſcribes its Surface.

DEF. V. An OBLIQUE CYLINDER, is when its Axe, is inclined to the Baſes, (ſee Fig. 2)

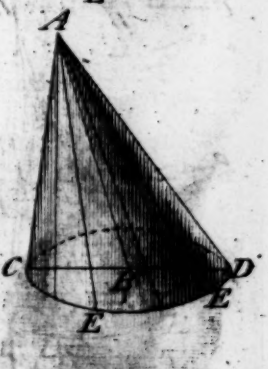
DEF. VI. A CONE is a SOLID, having but one Plane Surface, which is a Circle; and a convex Surface, returning into itſelf; continually varying, in convexity, from the Circle which bounds it, till it terminates in a Point. As ACD, or *ACD*.



DEF. VII. The BASE, of a CONE, is the circular Plane CD, or *CD*.

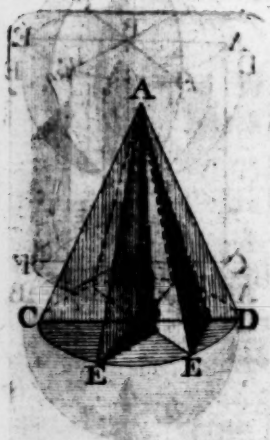
DEF. VIII. The VERTEX, of a CONE, is the Point A or *A*.

DEF. IX. The AXE, of a CONE, is the Right Line AB, or *AB*; from the Vertex A, or *A*, to the Center B, or *B*, of its Baſe.



N. B. A Cone may be conceived to be generated thus:

If CD or *CD* be a Circle, and any Point, A, or *A*, be taken out of its Plane; then, if a Right Line [illegible], be applied from the Point A, or *A*, to any Point C, or *C*; and being fixed, at A, or *A*, let it be revolved around the circumference of the Circle; in which revolution, it will deſcribe the Cone A C D, or *A C D*.

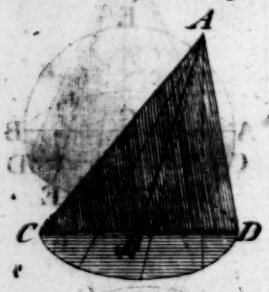


DEF. X. A RIGHT CONE is when the Axe (AB,) is perpendicular to the Plane of its Baſe.

A Right Cone may be conceived to be generated by the revolution of a right angled Triangle, ABC, on either Side containing the Right Angle, as AB, which remains fixed; and is therefore called its Axe or Axis, on which it revolves.

The other Side, BC, deſcribes the Baſe, in its revolution; whilſt the Hypothenuſe, AC, deſcribes the Surface of the Cone, and preſcribes its Bounds.

Every Section, by a Plane, through the Axe of a Right Cone, is an Iſoſceles Triangle, as CAD; CD is its Baſe; and a Right Line, AC, or AE, from the Vertex to any Point in the Circumference of its Baſe, is a Side of the Cone.



DEF. XI. A SCALENE or OBLIQUE CONE is when the Axe, *AB*, inclines to the Baſe, *CD*.

Every Section of an oblique Cone, through its Axe, is a Scalene Triangle; except in one poſition, when the Axe is at right angles with the Line of Section with the Baſe; and is, therefore, called a Scalene Cone.

When the Diameter of its Baſe, *CD*, is equal to the ſhorteſt Side *AD*; the Section, *CAD*, through that Side, perpendicular to the Baſe, is an Iſoſceles Triangle, and *AC*, the longeſt Side, is its Baſe; nevertheleſs *CAD* is called a Scalene Cone.

DEF. XII. A SPHERE is a SOLID, bounded by one uniform convex Surface, which has no Bounds.

As the Circle is bounded by one Line, its Circumference, ſo the Sphere is bounded by one Surface.

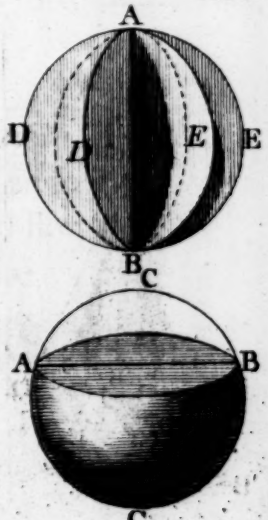
DEF. XIII. The CENTER of a SPHERE is the middle Point of the Solid; which is equally diſtant, every way, from its Surface. As C.

DEF. XIV.

DEF. XIV. The DIAMETER, of a SPHERE, is any Right Line paſſing through its Center, and terminated by its Surface. As A B.

N. B. A Sphere may be conceived to be generated, by the revolution of a Semicircle; as ADB, on its Diameter AB, which remains fixed whilſt the Semicircumference revolves; and, in its revolution, deſcribes, or rather preſcribes, the Bounds of the Sphere.

The Diameter, AB, being at reſt, on which if the Sphere revolved, it would appear at reſt, is therefore called an AXE, or Axis of the Sphere.



DEF. XV. A SEGMENT, of a SPHERE, is any portion, cut off by a Plane. As A C B.

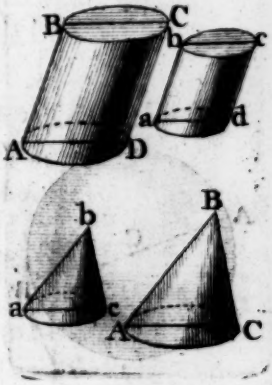
Its Baſe, AB, is a Circle, made by the Section.

DEF. XVI. A HEMISPHERE is a Segment of a Sphere. When the Plane of the ſection paſſes through its Center, each Segment is a Hemiſphere. As ADB, and AEB, made by the Plane AB.

DEF. XVII. SIMILAR CYLINDERS, and CONES, are ſuch as have their Axes, and the Diameters of their Baſes Proportionals.

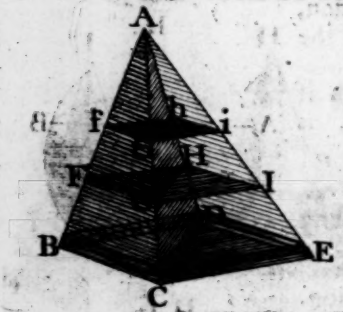
If they are ſcalene or oblique; Sections through the Axes of Cylinders, perpendicular to their Baſes, or equally inclined to their Baſes, are ſimilar Parallelograms; as ABCD, a b c d; and, in Cones, they are ſimilar Triangles; as A B C, a b c.

All Spheres are ſimilar to one another.



THEOREM I.

If a Pyramid be cut in two parts, by a Plane parallel to its Baſe; the leſſer Pyramid, made by that Section, will be ſimilar to the whole Pyramid.



Firſt; let A be the Vertex, and, BCD the Baſe of a triangular Pyramid; ABCD.

Let FGH be a Section made by a Plane, parallel to BCD.

I ſay, the Pyramid, AFGH, is ſimilar to the Pyramid ABCD.

Dem. Becauſe the Planes BCD, FGH are parallel, and they are both cut by the Plane BAC, the Sections, FG and BC, are parallel; for the ſame reaſon, the Sections GH and CD, FH and BD are parallel. - - Th. 8. 7.

Wh. AF:FB::AG:GC, and as AH:HD - - 2. 6.

conſ. AF:AB::AG:AC,&c.(6.5.) i.e. asFG:BC,&c-4.6.

and conſequently, the Triangle AFG is ſimilar to ABC, AGH to ACD, and AFH to ABD;

alſo, the Baſe, FGH, to the Baſe, BCD;

Therefore, the Pyramid AFGH is ſimilar to ABCD.

After the ſame manner, the Pyramid Afgh may be proved ſimilar to either of the other.

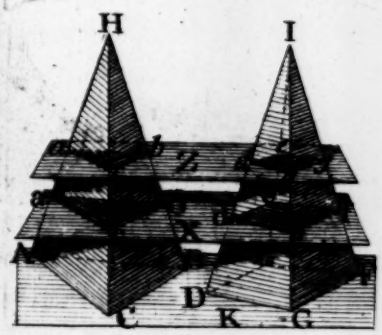
Again, Let CBDE be the Baſe of a quadrangular Pyramid, and, let GFHI be a Section, parallel to the Baſe.

It may be demonſtrated, in the ſame manner, that each Triangle AFG, AGI, &c. is ſimilar to ABC, ACE, &c. alſo, FGH is ſimilar to BCD, and GHI to CDE; conſequently, GFHI is ſimilar to CBDE. - 13. 6.

Wherefore, ſince a Pyramid, whoſe Baſe is a Poligon of any number of Sides, may be divided, by diagonal Sections, into triangular Pyramids, the ſame Demonſtration will hold true in every Pyramid whatever.

THEOREM II.

Pyramids, having equal Baſes and equal Altitudes, are equal. (See the laſt Figure.)



Firſt; let the Pyramids A B C D and C A D E have triangular Baſes, B C D and C D E, and a common Vertex, A; conſequently, they have equal Altitude. I ſay, they are equal Pyramids.

Let there be made ſeveral Sections, G F H I, g f h i, parallel to the Baſes, B C D, C D E, which are in one Plane.

Dem. Then, becauſe they are both cut by a Plane, parallel to their Baſes, and the Baſe BCD = CDE; FGH = GHI. For, FGH is ſimilar to BCD, and GHI to CDE; - Th. 1. and each has that Ratio to the other, reſpectively, which is duplicate of AC to AG; i. e. of CD to GH - 12. 6.

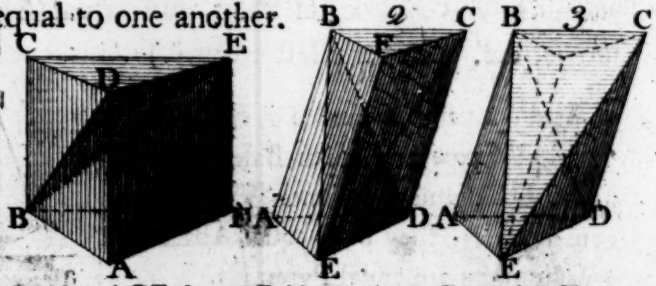
After the ſame manner, the Section f g h, may be proved equal to g h i; and conſ. every Section, made by Planes par. to CBDE, will be equal, each to the other, reſpectively. Therefore, the Priſm B A C D is equal to C A D E.

For, the Sections being equal, every where; conſ. if each Pyramid be conceived to be compoſed of an infinite number of Triangular Priſms, whoſe height (or thickneſs) are all equal, and the leaſt that can be conceived, laid on one another, as at BCD, CDE, being each eq. to the contiguous one, the whole is eq. to the whole.

Again. Suppoſe the Baſes of the Pyramids to be different in Figure, as ABC and DEFG; their Vertices, H and I, being equally diſtant from the Plane of their Baſes (AKF) conſequently, every Section, a b c, or *abc*, and d e f g or *defg* (made by the Planes X and Z, parallel to AKF) being in the ſame Ratio, each to the other, reſpectively, as the Baſes, are equal. Therefore, the Pyramid A H B C is equal to DEIFG.

THEOREM III.

Every Priſm, having a triangular Baſe, may be divided, into three Pyramids, having triangular Baſes, which are equal to one another.



First; let ACF be a Priſm, whoſe Baſe ABF, and conſequently CED, are right angled Iſoſceles Triangles.

Let the Planes, ABCD and ADEF, be Squares, and BAF, CDE, Right Angles; conſ. BCEF is a Rectangle, under the Side of a Square and its Diagonal (BC and CE).

Draw, AE, BE, and BD (Diagonals in each Face) and imagine Planes to paſs thro'. AE & BE, BD&BE-Ax.4.7.

The Pyramids, ABEF, AEDB, and BCED, made by the Sections of thoſe Planes, are equal and ſimilar.

DEM. The Triangle ADE = AEF, and ADB = BCD-15.1
But, ABCD and ADEF are Squares (by Suppoſition) and, they have a com. Side, AD; wh. they are equal-Ax.B.2
Alſo, BAF is a R. Angle (Sup.) conſ. CDE is a Rt. one; -5.7.
wh. the Tri's. AFB, CED, ADB, BCD, ADE, and AEF are all equal, and ſimilar to one another - 8. 1.
But, AEB and BED (the Planes of the Sections) are each common to two Pyramids; and they are equal and ſimilar to each other, and alſo to CEB, BEF. - - 7. 1.
For, AE and BD are each equal to CE and BF; being all Diagonals; alſo, AB, DE, CB, and EF are Sides of equal Squares, therefore equal; and BE is common to them all.
Wherefore, each Pyramid, ABEF, AEDB, and BCED, being contained by an equal number of Planes, equal and ſimilar to each other, reſpectively, are equal. - Ax. 5. 7.

For two Angles in each Pyramid, at A, C, D, and F, are contained by two R. Angles each, and half a R. Angle; as AFE, BFE, & AFB.

The remaining two, in each, meeting at B and E, are each contained by two half right Angles, (AEF, BEF, &c.) and an Angle AEB, equal DBE, (each common to two Pyramids) each equal CEB, or EBF; being contained by the Diagonal of a Square, (C E) and the Hypothenuſe (BE) of a right angled Triangle, under the Side and Diagonal, BC, and CE.

2nd. Let ABCE be a triangular Priſm; i. e. whoſe Baſes, ADE and BCF, are Triangles; ſuppoſe them Scalene, and the Sides AB, &c. of the other Faces, not perpendicular to them; the Priſm is conſ. oblique in every reſpect. The Priſm ABCE, may be divided into three eq. Pyramids.

Draw the Diagonals, BE, BD, & EC; and ſuppoſe a Plane to paſs thro'. BE & BD, and, another, thro'. BE & EC - Ax. 4. 7

The Priſm will be divided into three Pyramids; ABDE, BCEF, and BECD.

I ſay, thoſe Pyramids are equal to one another.

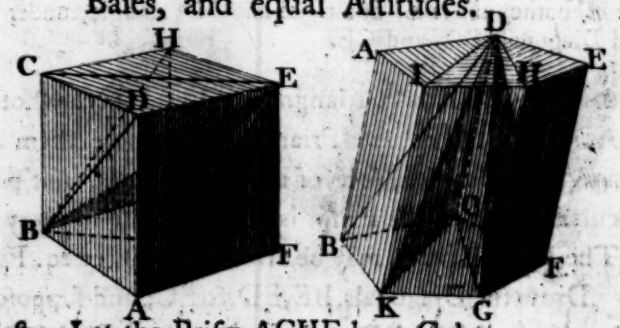
Dem. The Pyramids, ABDE and BCEF, have equal and ſimilar Baſes, ADE & BCF, and equal Altitudes; Def. 8. 7. Conſequently, every Section, made by parallel Planes, at equal diſtances from their Baſes, are equal, and alſo ſimilar; Therefore, the Pyramids are equal. - - Th. 2.

Again. Suppoſe the Pyr. BCEF (eq. ABDE) taken away; the Face BCE is common to the Pyramid BECD (ſee Fig. 3) Then, in the Pyramids ABDE, BECD; becauſe ABCD is a Parallelogram (Def. 7.) it is divided by the Diagonal, BD, into congruous Triangles, ABD and BCD. - 15. 1. Let them be conſidered as the Baſes of the two Pyramids; E being their com. Vertex, they have, conſ. equal Altitude; wh. every Section, parallel to ABCD, will be equal, and alſo ſimilar; and conſequently, the Pyramids are equal - Th. 2.

If ABDE be ſuppoſed removed (Fig. 2.) then the Pyramids BCEF, & BECD are equal; for their Baſes CEF, & CED are equal (15. 1.) and B is their common Vertex; Therefore, they are all equal to one another. Q. E. D.

THEOREM IV.

Every Prism is triple of a Pyramid, having equal Bases, and equal Altitudes.



First. Let the Prism ACHF be a Cube.

Then, a Plane drawn through the opposite Diagonals CE and BF, will cut the Cube into two equal triangular Prisms; whose Faces FD and AC, &c. are Squares, and the Angles BAF, CDE, CHE, and BGF, are R. Angles.

Draw, AE, EB, and DB; dividing the hither Prism, ABCEF, into three equal and similar Pyramids ABEF, AEDB and BCED, (as in Case 1st of the 3rd.)

Also, draw EG. BEFG is a Pyramid, equal and similar to ABEF; and the Prism ACHF is triple of the Pyramid ABEFG.

DEM. For, the Base BFG is equal and similar to ABF - 15. 1.
and the Face BEG is congruous with AEB; - Ax. 7. 1.
(for EG bisects the Square FH, and AE bisects FD)
But, BEF is common; and BEG, AEB are congruous - 7. 1.
(for, BG is equal to AB, EG=AE, and BE is common)
wh. the Solid, BEFG, is contained within four triangular Planes, equal and similar to those of AEBF;
and consequently, they are equal Pyramids. - Ax. 5. 7.

But, the triangular Pyramid, AEBF, is equal one third of the Prism, ABCEF; and BEFG is equal one third of BHF, equal CAE. - - - - Cor. 1. 3,

Therefore, the quadrangular Pyramid, ABEFG, is equal one third of the Cube ACE; or, the Cube triple of the Pyramid, on the ſame Baſe, ABGF, and having the ſame Altitude, EF.

2nd. After the ſame manner, as in Caſe II. of the laſt, every triangular Priſm may be proved triple of a Pyramid, having the ſame, or an equal Baſe and Altitude.

Wherefore, ſince every Priſm, having a quadrangular Baſe, may be divided into two triangular Priſms; and each triangular Priſm, into three equal Pyramids having triangular Baſes; conſ. a Pyramid on the whole quadrangular Baſe of the Priſm, and having equal Altitude, is equal one third of the Priſm, or the Priſm triple of the Pyramid.

3rd. Let the Baſe, AIHED, of the Priſm, AKFD, be a Pentagon; and let the Diagonals, CG, CK, DH, DI, be drawn.

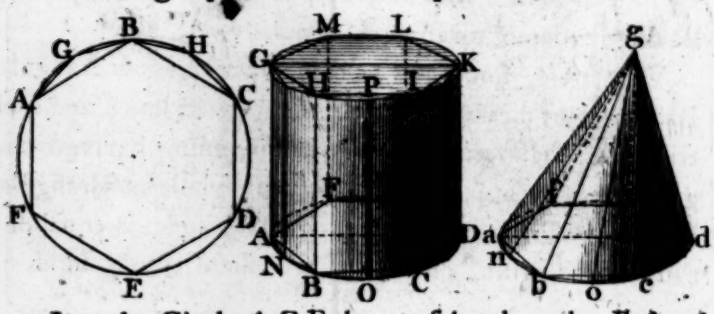
Then, becauſe CD is parallel to AB and EF; and, GH is parallel to EF, and IK to AB - - Def. 33.
DC is parallel to IK and GH. - - Th. 4. 7.
wh. Planes may paſs through DC & GH, DC & IK - Ax. 2.
and conſ. the Priſm, AKFD, will be divided into three triangular Priſms, by the Planes, DCKI and DCGH; viz. AKD, KDG, and DGE.

Draw the Diagonals BD, DF, DG, and DK; BDFGK is a Pyramid having the ſame Baſe, BCFGK, and Altitude as the Priſm.

But, BDCK is a Pyramid, having the ſame Baſe, BCK, and Altitude, as the Priſm AKD;
AKD is therefore triple of the Pyramid, BDCK.
For, the ſame reaſon, KIDHG is triple of the Pyr. KDGC;
and the Priſm DGE is triple of the Pyramid CDFG.
But, the Pyramid BDFGK is equal to the three Pyramids BDCK, KDGC, and CDFG; and the Priſm AKFD is equal to the three Priſms AKD, KIDHG, and DGE.
Th. the Priſm AKFD is triple the Pyr. BDFGK. Q.E.D.

THEOREM V.

Every Cone is equal to the third part of a Cylinder, having equal Baſes and equal Altitudes.



Let the Circle A C E be conſidered as the Baſe of a Cylinder or Cone; and, let a regular Poligon, ABCDEF, of any number of Sides be inſcribed; it is manifeſt, that the Poligon is leſs than the Circle.

Let the Arks, AGB, &c. be biſected, as at G; and draw AG and BG, &c. This Poligon will be greater than the firſt, but it is alſo leſs than the Circle.

Wherefore, it is evident, that, the more the Sides of the Poligon are multiplied, the nearer it approaches to the Circle, till the difference is leſs than any aſſignable Quantity whatever. The difference is the Segments AG, GB, &c.

And, by the ſame reaſoning, every Poligon, circumſcribed, will be greater than the Circle; conſequently, they will at laſt end in the Circle; i. e. the Perimeter of the Poligon, in the Circumference of the Circle, and their Areas will be equal, or the difference will be leſs than any other Quantity.

Let A G K D be a Cylinder whoſe Baſe is A D; and a g d a Cone, whoſe Baſe, a d, is equal to the Baſe AD, of the Cylinder; and let them alſo have equal Altitudes.

I ſay, the Cylinder AGKD, is triple of the Cone, a g d.

DEM.

DEM. Let there be regular Poligons, of any number of Sides, inſcribed; as A B C D E F, and a b c d e f, whoſe Sides AB, BC, &c. a b, b c, &c. are equal; conſequently, they are equal Poligons.

Let a Priſm be conſtructed on the Poligon, ABCDEF, whoſe Sides AG, BH, CI, &c. are in the Surface of the Cylinder; and from every Angle of the Poligon, a b d, let there be drawn Right Lines a g, b g, &c.

Then, a b g d c is a Pyramid, whoſe Baſe is equal to the Baſe of the Priſm (by Conſtruction); and, having equal Altitudes, the Priſm, AMKC, inſcribed in the Cylinder, is triple of the Pyramid, inſcribed in the Cone. - Th. 4.

Again. Let the Arks AB, BC, &c. be biſected, at N and O; and let a Priſm be inſcribed, having twice the number of Sides.

Alſo, let the Pyramid, inſcribed in the Cone, have the ſame number of Sides, as the Priſm; their Baſes are equal; and, conſequently, the Priſm is triple of the Pyramid - 4.

And ſo it muſt always be; till, by multiplying the Sides of the Priſm, its Surface will be the ſame as the Surface of the Cylinder, and the Surface of the Pyramid as the Cone; the Baſes of the Priſm and Pyramid will be the ſame as the Baſes of the Cylinder and Cone.

But, they are equal, and they have equal Altitudes.

Therefore, the Cylinder is triple of the Cone; or, the Cone equal to one third of the Cylinder. Q. E. D.

THEOREM VI.

Prisms, or Pyramids, having equal Altitudes, have that Ratio to each other, as their Bases.

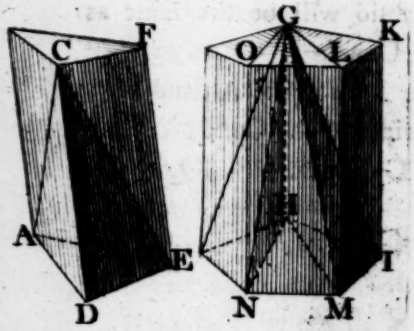
First. Let the Prisms have quadrangular Bases.

If they are Parallelopipeds, it is already proved - Cor. 1. 17. 7.

For, every Parallelopiped (is a Prism; Def. 8. 7.) may be divided, by a Plane passing through its opposite Diagonals, into two equal and similar triangular Prisms; consequently their Bases and Altitudes are equal. - - - - - - Th. 16. 7.

And it is manifest, that if the Base of one be double, or triple, of the other, seeing they have equal Altitudes, they are as their Bases; i. e. the Prism, whose Base is double or triple of the others Base, is double or triple of the other Prism; and consequently, whatever ratio one Base has to the other, the Prisms have necessarily the same. (For, whether the Bases be similar Figures or not, being equal, it is the same; as it is proved, in Case the 3rd of the 17th, being Parallelopipeds.)

Also, if their Bases be equal, they are as their Altitudes - Cor. 2. 17. and, if the Prisms are equal, their Bases and Altitudes reciprocate, (Th. 19.) Consequently, every triangular Prism is to another, of equal Altitude, in the same ratio as their Bases; seeing, they are each half a Parallelopiped or Prism, whose Base is double the Base of the triangular Prism, and having equal Altitudes.



Let the Prisms AFD and EGKM have equal Altitudes; the one on a triangular Base, the other pentagonal.

Let the Diagonals GL, GO, HM, and HN be drawn, between opposite Angles in the equal Pentagons; and imagine Planes OGHN, and GLMH, drawn through those Diagonals, dividing the Prism, EGKM, into three triangular Prisms, EGN, NGM, and MGI.

DEM. Then; because the triangular Prisms, AFD and EGN, have equal Altitudes, they have that Ratio to each other, as their Bases ADE to EHN. - proved above.

i. e. the Priſm AFD:EGN:: ADE:EHN }
And, the Priſm AFD:NGM:: ADE:NHM } their Baſes.
alſo, the Priſm AFD:MGI :: ADE:MHI }
Wh. as the Baſe ADE:EHN+NHM+MHI }
ſo is the Priſm AFD:EGN+NGM+MGI } - 2. 5.
i.e. as the Baſe ADE: EHIMN :: the Priſm AFD: EGKM.

2nd. Becauſe the Priſm AFD is triple the Pyramid ACED; and the Priſm EGKM is triple of the Pyramid NEGIM-4.
Alſo, as Baſe ADE:EHIMN,:: the Priſm AFD:EGKM
conſ. as Baſe ADE:EHIMN,:: the Pyr. ACED:NEGIM

Cor. 1. Cylinders, having equal Altitudes, have the ſame Ratio to each other as their Baſes.

For, ſimilar Poligons, inſcribed in Circles, are, to each other, as the Squares of their Diametens. - - - Th. 14. 6.
And it has been proved, that all Priſms, having equal Altitudes, are in the Ratio of their Baſes, to each other; conſequently, Cylinders are to each other in the ſame Ratio.
Becauſe, every Cylinder, whether it be right or oblique, is equal to a Priſm, whoſe Baſe is the ſame, or equal to the Baſe of the Cylinder; i. e. having its Sides multiplied continually till it ends in the Surface of the Cylinder; and its Baſe, in the Baſe of the Cylinder.

Cor. 2. Cones, having equal Altitudes, are to each other, in the Ratio of their Baſes.

Becauſe, a Cone is equal to one third of a Cylinder, having equal Baſes and Altitudes; and Cylinders are in that Ratio to each other; therefore Cones have the ſame Ratio as Cylinders.

Cor. 3. Priſms and Pyramids, having equal Baſes, are to each other as their Altitudes.

Becauſe Parallelopipeds are Priſms; and it has been proved, that Parallelopipeds are to each other in that Ratio; and conſequently, all Priſms, having equal Baſes, are as their Altitudes.

Cor. 4. Cylinders and Cones, having equal Baſes, are to each other as their Altitudes.

Becauſe, there is the ſame Ratio between Cylinders and Cones, as between Priſms and Pyramids, having equal Baſes. Therefore, &c.

THEOREM VII.

Similar Priſms and Pyramids are, to one another, in the triplicate Ratio of their correſponding Sides. *

In reſpect of Priſms (being alſo Parallelopipeds) this property is already proved (in Theo. 18th of the 7th) and conſequently, in Priſms having triangular Baſes; ſeeing, every ſuch Priſm is half a Parallelopiped, whoſe Baſe is double of the Priſm and having equal Altitudes.

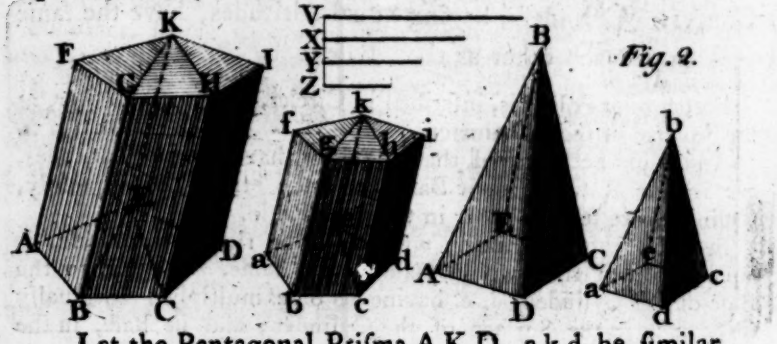


Fig. 2.

Let the Pentagonal Priſms A K D, a k d be ſimilar.

Take any Right Line, V; and as a b is to A B, &c. make X to V; and as X is to V, ſo make Y to X; alſo, as Y is to X, or X to V, make Z to Y.

I ſay, that, as V is to Z, ſo is the Priſm AKD to akd.

Let them be divided by diagonal Planes, BK, KC, bk, kc, paſſing through the oppoſite Diagonals; dividing the Priſms into three triangular Priſms, each.

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* I cannot conceive the reaſon, why Euclid confines this proportion, and the following, to Pyramids only; and, to ſuch, only, as have triangular Baſes; ſeeing that, it holds equally true in Priſms; and alſo in Pyramids, having poligonal Baſes, of any kind.

DEM.

DEM. The Poligons ABCDE, abcde, are ſimilar - Hyp. wh. the Triangles ABE, abe, &c. are ſimilar. - 13. 6. and conſ. the Priſms AKB, akb, &c. are ſimilar - Ax. 11. 7 (For, as AB:ab, or as BC:bc, ::BG:bg, and GK:gk; conſequently, the diagonal Planes, BK and bk, KC and kc, are ſimilar)

Wherefore, ſince each triangular Priſm is equal to half a Parallelopiped, whoſe Baſe is double of the triangular Priſm, and Altitudes equal (above) they have that ratio to each other, which is triplicate of their correſponding Sides, i. e. as V to Z. (Ax. 8. 5.) - - - Th. 18. 7.
But the Priſm AKB:akb::BKC:bkc, and as CKD:ckd. wh. as AKB:akb::AKB+BKC+CKD:akb+bkc+ckd.*
i. e. as the Priſm AKB : akb :: the Priſm AKD : akd.
But, AKB:akb::V:Z, Th. the Priſm AKD:akd::V:Z.†

2nd. Let the Pyramids ABCD, abcd be ſimilar. (Fig. 2.)
The Pyramids ABCD, abcd, being ſimilar, are each equal to the third part of a Priſm on the ſame Baſe and Altitude.

But, Quantities are in the ſame Ratio, to each other, as their Equimultiples or equal Parts. - - Ax. 8. 5.
and the triples of the Pyramids ABCD, abcd, are as V to Z.
therefore, the Pyramid ABCD : abcd ::V:Z. Q. E. D.

COR. Similar Cylinders are, to one another, in the triplicate Ratio of their Diameters or Sides.

For, they are in the ſame Ratio, to each other, as ſimilar Priſms inſcribed; or, as Priſms, whoſe Baſes are equal to the Baſes of the Cylinders, and having equal Altitudes.

COR. 2. Similar Cones are, to each other, in the triplicate Ratio of the Diameters of their Baſes, or of their Axes.

Becauſe a Cone is equal to the third part of a Cylinder, on equal Baſes and Altitudes.

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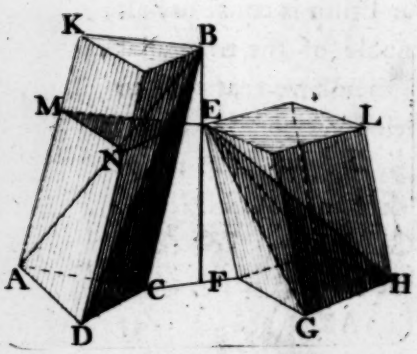
* By Theo. 2nd of the 5th. † By Axiom 13 of the 5th.

THE O-

THEOREM VIII.

Equal Priſms, or Pyramids, have their Baſes and Altitudes reciprocally proportional.



If the Priſms are Parallelopipeds it is already proved. - - 19. 7.

Let the triangular Priſm AKC be equal to the quadrangular Priſm FLG; whoſe Baſes, ACD, FGHI, are in the ſame Plane; to which, Draw the Perpendicular BF. BF and EF are their Altitudes.

I ſay, the Baſe, ADC, is to the Baſe, FGHI, as EF is to BF.

Let the Plane of the Top, LE, be produced, cuting the triangular Priſm, at EMN.

DEM. The Priſm AMC:AKC::EF:BF, their Altitudes. And - - - AMC:FLG::ADC:FGHI, their Baſes. But, the Priſm FLG is equal to AKC; by Hypotheſis, conſequently, AMC:FLG::EF:BF, i.e. as ADC:FGHI Therefore, as ADC:FGHI :: EF : BF. Q. E. D.

2nd. Draw the Diagonals AB, DB; EG, EH, and EI.

The Pyramid ABCD is equal to one third of AKC - 4. And, the Pyramid FGEHI = one third part of FLG. But, Quantities are in the ſame Ratio as their Equimultiples; wherefore, the the Pyramid ABCD is equal to FGEHI.

But, they have the ſame Baſes and Altitudes, as the Priſms, AKC and FLG.

Th. their Baſes and Altitudes are reciprocally proportional.

COR.

Cor. Equal Cylinders, or Cones, have their Baſes and Altitudes reciprocally Proportional.

Becauſe, Cylinders are equal to Priſms whoſe Baſes and Altitudes are equal; and Cones to Pyramids.

In theſe eight Theorems, and their Corollaries, with the Corollary to the next, is contained the whole of Euclid's 12th Book; thoſe which follow, are ſelect Theorems from Archimedes; which ſhews how the ſurface of the Sphere may be aſcertained: alſo, of any Segment, or Portion of its Surface, intercepted between parallel Circles, with great accuracy.

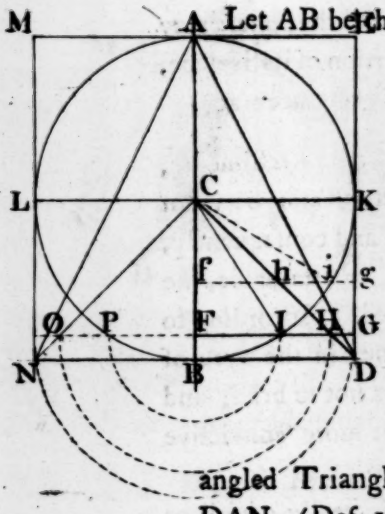
The next Theorem, which is not from Archimedes, ſhews, in a brief and elegant manner, the Ratio between the Sphere and a circumſcribing Cylinder; and conſequently, if the quantity of the Cylinder be known, or attainable, the quantity of the Sphere may alſo be obtained. According to Archimedes, it is determined by the Cone, in the 17th of this Book; which, though perfectly true, is not ſo brief; and though perhaps more geometrical, yet not more convictive and ſatisfactory.

In the following is contained all that is really uſeful or valuable, in reſpect of the Cylinder, Cone, and Sphere, nor is it poſſible, I think, to obtain the Area of the Surface of a Sphere in a more conciſe manner.

In ſhort, without the following Theory, I cannot but think the Doctrine of Solids imperfect; ſeeing that, the Quantity of a Sphere, the moſt perfect Solid, cannot, by Euclid, be obtained, in reſpect either of its Superficies or ſolid Contents; for want of which, Artificers are frequently at a loſs, in meaſuring a Dome or other ſpherical Object.

THEOREM IX.

Every Sphere is equal to two thirds of a circumſcribing Cylinder; i. e. of a Cylinder whoſe Diameter and Altitude are equal, each to the Diameter of the Sphere.



Let AB be the Diameter of a Sphere; biſect AB, at C.

On C, deſcribe the Semicircle AKB; and, on AB, conſtruct the Rectangle AEDB, circumſcribing it, and draw AD.

Then (AB remaining fixed) imagine them all revolved about AB, as an Axis; the Semicircle, AKB will deſcribe a Sphere, AKBL, (Def. 14.)

The Rectangle, AEDB, will deſcribe a Right Cylinder, DEMN, containing the Sphere (Def. 4.) and, the right-angled Triangle, ADB, will generate a Right Cone, DAN, (Def. 10) whoſe Baſe, DN, is the ſame as the Cylinder, and equal to the Diameter of the Sphere.

The Cone, DAN, is equal to one third part of the Cylinder. - - - - - - Th. 5.

I ſay, that the Sphere, AKBL, is equal to the remaining two thirds.

Draw CD; and FG (at pleaſure) parallel to CK; cuting CD at H, and the Circumference of the Sphere at I; and draw CI.

DEM. In the Rt. angled Tri. CFI, $CI\square = CF\square + FI\square$ - 20. 1 and, the Areas of Circles are as the Squares of their Diameters. - - - - - - C. 1. 14. 6. wherefore, a Circle, deſcribed on CI, is equal to two Circles, on CF and FI.

Now, in the revolution of the Square CKDB, there will be deſcribed the Cylinder KDNL; the Ark BIK will deſcribe a Hemiſphere; and, CD will deſcribe the Surface of a Right Cone, DCN; alſo, the Points G, H, and I, will deſcribe Circles, with the ſeveral Radii FG, FH, and FI; FG is equal to CI (equal CK) and FH is equal to CF. - - - - 9. 1.
(for, the Angle CFH is Right, and the Angles FCH, FHC, are half Right; ſeeing that, the Diagonal, CD, biſects the Angles BCK, KDB)

Wherefore, the Circle deſcribed by FH (equal CF) is equal to the difference between the Circles deſcribed by FG (equal CI) and FI.

Conſequently, the Circle, deſcribed by FH, is equal to the Annulus or Ring, deſcribed by GI;

and, conſequently, every Section (FG or fg) of the Cone DCN, by a Plane parallel to its Baſe, is equal to a Section of the concave Solid, remaining when the Sphere is ſuppoſed to be taken out of the Cylinder.

Wherefore, the Cone, DCN, is equal to the Solid deſcribed by the mixed Triangle KiIBD.

But, the Cone, DCN = one third of the Cylinder KDNL conſ. the concave Solid (taking away the Hemiſphere, KBL, out of the Cylinder, KDNL) is equal to one third of that Cylinder; and conſequently, the Hemiſphere, KBL, is equal to two thirds of the Cylinder.

But, the Cone, DCN, is half the Cone DAN; and, the Cylinder, DKLN, is half the Cylinder DEMN; C. 4. 6.
alſo, the Sphere, AKBL, is double the Hemiſphere, KBL.
Therefore, the Sphere AKBL, whoſe Diameter, AB or KL, is equal to the Diameter and Altitude of the Cylinder, DEMN, is equal to two thirds of the Cylinder. Q. E. D.

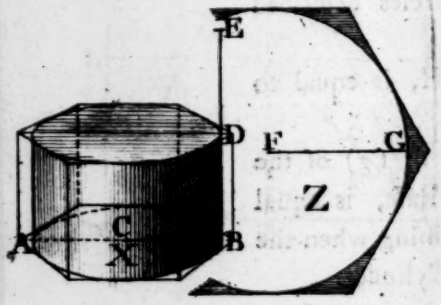
COROLLARY. The 18th Propoſition, of Euclid.

Spheres have that Proportion to one another, which is triplicate of their Diameters;

For, all Cylinders, being ſimilar, are in that Ratio to one another; and conſequently, Spheres, inſcribed in Right Cylinders whoſe Diameters and Altitudes are equal, reſpectively, to the Diameter of the Spheres, being two thirds of ſuch Cylinders, are in the ſame Ratio to each other as the Cylinders; i. e. in the triplicate Ratio of their Diameters.

THEOREM X.

A Circle, whoſe Radius is a mean Proportional between the Diameter of the Baſe, and the Side of a Right Cylinder, is equal to the cylindrical Surface.



Let AB be the Diameter of the Baſe of the Cylinder AD, and let BD be its Side.

Take FG a mean Proportional, between AB and BD, (Prob. 30) and on FG, deſcribe a Circle; let C be the Center of the Baſe of the Cylinder; produce BD, and make DE equal to BD.

Then, CB, FG, & BE are Proportionals; for, CB : AB :: BD : BE, - - - Ax. 8. 5. wherefore, CB × BE = AB × BD - - 9. 6. i. e. to FG □; for AB × BD = FG □. - Cor. to 9. 6.

Let ſimilar Poligons (X and Z) be circumſcribed about the Baſe of the Cylinder, and the Circle whoſe Radius is FG; and let a Priſm, be ſuppoſed to circumſcribe the Cylinder.

DEM. A Triangle, whoſe Baſe is equal to the Perimeter of the Poligon, and its Altitude equal CB, is equal to the Poligon;* and a Rectangle under the Perimeter and BD, a Side of the Priſm, is equal to the Superfices of the Priſm-1.2 Alſo, a Triangle under the Perimeter and BE is equal to a Rectangle under the Perimeter and BD, half BE.

* This follows from the 18th of the firſt. For, all the Sides of the Poligon are equal; and all Triangles having equal Baſes and Altitudes are equal. (See Art. 7. P. 7. in the Appendix).

But, a Triangle, under the Perimeter and C B, is to a Triangle under the ſame Perimeter & BE, as CB is to BE-1.6 conſ. the Poligon is to the Surface of the Priſm, as CB:BE. But, CB to BE is duplicate of CB to FG - Def.12.5. wherefore, the Poligon X is to the Superficies of the Priſm, in the Duplicate ratio of CB to FG.

But, the Poligon X is to Z in the Duplicate ratio of their Sides, or Perpendiculars, CB to FG, i. e. as CB to BE. wh. the Poligon X has the ſame Ratio to Z, as X has to the Superficies of the Priſm, AD; conſ. they are equal-Ax. 5.5

After the ſame manner, it may be proved that the Superficies of any other Priſm, whoſe Faces being multiplied infinitely, till they end in the Surface of the Cylinder, AD, is equal to the Poligon circumſcribing the Circle, Z, whoſe Sides being multiplied, end at laſt in the Circle.

Therefore, the Circle, Z, whoſe Radius is a Mean, between AB & BD, is equal to the Surface of the Cylinder, AD.

Cor. 1. The Superficies of a Right Cylinder is to its Baſe, as the Side of the Cylinder to the fourth part of the Diameter of the Baſe.

For the Superficies of the Cylinder is equal to the Circle, whoſe Radius is FG; and that Circle, is to the Baſe of the Cylinder in the duplicate ratio of F G to C B; i. e. of B E to C B; i. e. of B D to half C B.

Cor. 2. The Superficies of a Cylinder whoſe Side is equal to the Diameter of its Baſe, is equal to four times the Area of the Baſe; conſ. if the Side be a fourth part of the Diameter the cylindrical Superficies, and the Baſe are equal.

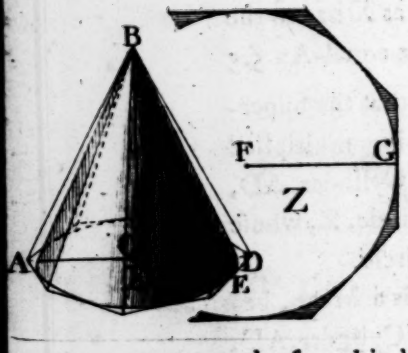
Cor. 3. The Superficies of a Right Cylinder is equal to a Rectangle, under its Circumference and Side.

Cor. 4. Cylindrical Surfaces, of equal height, are to each other as the Diameters of their Baſes; and, having equal Baſes, they are as their Altitudes.

Cor. 5. The Surfaces of ſimilar Cylinders have that Ratio between them; which is duplicate of their Diameters.

THEOREM XI.

A Circle, whoſe Radius is a mean Proportional between the Side of a Right Cone and the Radius of the Baſe, is equal to the Superficies of the Cone.



Let ABD be a Right Cone, the Diameter of whoſe Baſe is AD, and C its Center.

Take FG, a mean Proportional between the Side, BD, and the Radius CD (equal BE and CE) and on FG deſcribe a Circle.

Let regular Poligons (X and Z) be circumſcribed about the Baſe of the Cone and the Circle, whoſe Radius is FG, of the ſame kind; and ſuppoſe a Pyramid, on the Poligon X circumſcribing the Circle.

DEM. Becauſe CE : FG :: FG : BE; by Hypotheſis.
CE to BE is duplicate of CE to FG. - Def. 12. 5.
But, as CE is to BE, ſo is the Triangle, whoſe Altitude is CE, and its Baſe, the Perimeter of the Poligon X, to the Triangle under BE and the ſame Perimeter. -Th. 1. 6.
Alſo, as CE is to BE ſo is the Triangle under CE and the Perimeter of the Poligon X, to a Triangle, under FG and the Perimeter of Z. - - Th. 6. and 12. 6.

But, a Triangle, under CE and the Perimeter of X, is equal to the Poligon X; and, the Triangle under FG and the Perimeter of Z is equal to that Poligon; alſo, the Triangle under BE and the Perimeter of X, is equal to the Superficies of the Pyramid.*

* This alſo follows from the 18th of the firſt Book of Elements, and Ax. 1. 1. (See the Reference to the laſt).

Wherefore,

Wherefore, the Poligon X has the ſame Ratio to the Poligon Z, and to the Superficies of the Pyramid; conſequently, the Poligon Z is equal to the Superficies of the Pyramid. - - - - - - - - Ax. 5. 5.

After the ſame manner it may be proved, that the Superficies of any other Pyramid (whoſe Baſe is a regular Poligon, ſimilar, to one circumſcribing the Circle, whoſe Radius is F G, till by multiplying the Sides, equally, the Perimeters, of both, fall into the Circumferences of the Circles, and the Superficies of the Pyramid into the Cone) is equal to the Poligon Z; conſequently, the Superficies of the Cone is equal to the Circle whoſe Radius is FG, a mean Proportional between the Side BD & the Radius CD.

Cor. 1. The Superficies of a Right Cone is to its Baſe, as the Side of the Cone to the Radius of the Baſe.

For, they are, reſpectively, equal to Triangles, whoſe Baſes are equal to the Circumference of the Baſe, and Altitudes equal to BD and C D reſpectively; conſequently, they are to each other, as B E to C E. - - - - - - - 1. 6.

Cor. 2. The Superficies of a Right Cone is equal to a Triangle, whoſe Baſe is equal to the Circumference of the Baſe of the Cone, and its Altitude equal to a Side of the Cone.

Hence it is manifeſt, that the Surfaces of Right Cones have all the ſame Properties as Triangles.

If their Sides be equal, they are, to each other, as the Diameters of their Baſes; and having equal Baſes, they are as their Sides; or as their Axes.

If they are ſimilar, the Ratio between their Surfaces is duplicate of their Sides, or Diameters of their Baſes.

If they are equal, their Sides and Diameters are reciprocally Proportional.

Cor. 3.

COR. 3. The Superficies of a Right Cylinder, is to a Right Cone, having equal Baſes and Altitudes, as the Side of the Cylinder, to half the Side of the Cone.

For, the Superficies of the Cone is to its Baſe, as the Side to the Radius; or, as half the Side to half the Radius. And the Superficies of the Cylinder, to its Baſe, is as the Side to half the Radius; therefore, the Surface of the Cone is to the Surface of the Cylinder, as the Side of the Cylinder to half the Side of the Cone.

COR. 4. An Equilateral Cone, i. e. whoſe Side is equal to the Diameter of its Baſe, has it Superficies double of its Baſe; and if the Vertex be right angled, the Ratio is as the Diagonal of a Square to its Side.

THEOREM XII.

If a Right Cone be cut by a Plane, parallel to its Baſe; a Circle whoſe Radius is a mean Proportional, between part of the Side (betwixt the Baſe and the Section) and the Radius of that Section added to the Radius of the Baſe, is equal to the conical Superficies, which is between the Section and the Baſe.

Let BAF be a Right Cone; and let DJ be a Section, parallel to the Baſe, BF.

Take GH a mean Proportional, between AB, and BC; and GI a Mean between AD and DE; on which, deſcribe Circles, X and Z.

The Circle X, on GH, is equal to the whole Superficies; and, that on GI = to the Surface of the Cone, DAJ - 11. and, AB × BC = AD × DE, + DB × BC, + DB × DE.*

Take GK a Mean, between DB and DE + BC; on which, deſcribe a Circle; let it be called Y.

I ſay, that Circle (Y) is equal to the conical Superfices BDJF; between the parallel Circles.

DEM. Becauſe GH is a Mean, between AB, & BC - by Con. the Square of GH = AB × BC; i. e. to the Rect. AB cd;
For the ſame reaſon, GI □ = AD × DE; i. e. to AD ea } 9.6
And, becauſe GK is a Mean, betwixt BD and DE + BC;
GK □ = BD × BC, + BD × DE; (equal Bf + Be)
i. e. to a Rectangle under BD and DE + BC.
Conſequently, GH □ = GI □ + GK □.
(Becauſe, the Rectangle equal to GH Square, is equal to two Rectangles, which are reſpectively equal to the two Squares, of GI and GK)

But, Circles are to each other, as the Squares of their Diameters, and conſequently as their Radii - C. 1. 14. 6.
And the Circle X is equal to the whole conical Surface; alſo, the Circle Z is equal to the Surface DAJ.
Therefore, the Circle Y (being equal to the Difference between the two, X and Z) is equal to the Superficies BDFJ; between the Section, DJ, and the Baſe.

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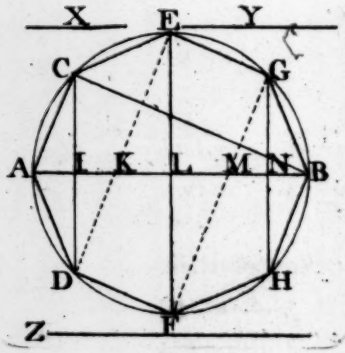
* This is deducible from the ſecond Book of Elements; having formed a Conſtruction.

Let ABcd be a Rectangle, under AB and BC; Ac being drawn, and Df parallel to Bc, cuting it at e; and ab (through e) parallel to AB.

Then, the whole, Bd (equal AB × BC) = Da (equal AD × DE) + BF (equal BD × BC) + ed (equal BD × DE, i. e. Be, 19. 1.)

THEOREM XIII.

If a regular Poligon, having an equal number of Sides, be inſcribed in a Circle, and if a Diameter (which is alſo a Diagonal of the Poligon) be drawn, and another Diagonal, from either extreme of the Diameter to the adjacent Angle; the Rectangle, under the Diameter and that Chord, is equal to a Rectangle under a Side of the Poligon and all the parallel Chords, joining two and two contiguous Angles.



In the Circle AGF, deſcribe a regular Octagon, ACEGBHFD; draw the Diameter AB, and the Diagonal BC; alſo, draw CD, EF, and GH.

I ſay, the Rectangle, under AB and BC, is equal to a Rectangle under AC (or AD) and CD, added to EF, added to GH.

Join DE and FG.

DEM. Becauſe CE = DF, and EG to FH, the Chords CD, EF, and GH, are all parallel - Cor. to 10. 3. And, for the ſame reaſon, AC, DE, FG, & HB are parallel; conſ. the Triangles ACI, IDK, KEL, &c. are ſimilar. Wh. AI:IC::IK:ID,::KL:LE,::LM:LF, &c. - Th. 4. 6. i. e. as AI : IC :: AB : CD + EF + GH, - 2. 5. for, the Diameter, AB, = AI + IK + KL, &c. and CD + EF + GH = CI + ID + EL, &c. - Ax. 1. 1. But, as AI:IC::AC:CB (7. 6) for ACB is a R. Angle - 12. 3. Conſequently, AC:CB::AB:CD+EF+GH - Ax. 13. 5. Therefore, AC × CD+EF+GH = CB × AB - Th. 9. 6.

COR. Hence it is manifeſt, that if in any Segment of a Circle, as GAH, a Poligon be inſcribed, on GH, whoſe Sides are equal, and equal in Number; a Rectangle, under the Chord CB and AN (part of a Diameter, perpendicular to GH) is equal to a Rectangle under a Side AC, and the Chords, CD + EF + GN, half the Baſe.

THEOREM XIV.

If a regular Poligon be inſcribed in a Circle, (as in the foregoing Theorem) and Chord Lines be drawn, (as there deſcribed); a Circle, whoſe Radius is a mean Proportional between the Diameter (AB) and the Chord CB, is equal to all the conical Superficies deſcribed by the revolution of the Poligon, on the Diameter or Axis, AB.

The ſame Conſtruction remaining, as in the former Figure.

Find X a mean Proportional between AC and CI; alſo Y, a Mean, between CE and CI added to EL.

Dem. Now, AC, CE, EG, & GB are equal, by Conſtruction.

And, AB×BC = AC×CD+EF+GH (Th. 13.) i. e. to AC×CI,+CE×CI+EL,+EG×EL+GN,+GB×GN

Find Z, a mean Proportional between AB and BC; the Square of Z is equal to the Squares of X and Y, taken twice; i. e. to AC×CD, +CE×EF, +EG×GH - 13.

On X and Y deſcribe Circles; the Circle X is equal to the Surface of the Cone DAC, - - - Th. 11.

and, the Circle Y is equal to the conical Surface ECDF - 12.

But, Circles are in the ſame Ratio as the Squares of their Diameters, (C. 1. 14. 6.) wherefore, a Circle, deſcribed on Z, is equal to X and Y, taken twice.

But, the Cone DAC = GBH; and ECDF = EGHF; for, CD is equal to GH, and CE to EG, &c.

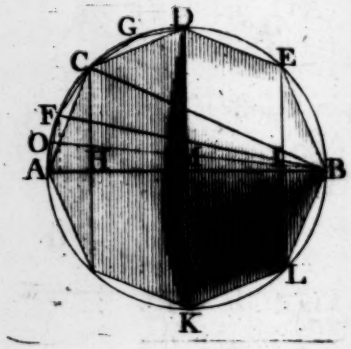
Therefore, a Circle (Z) whoſe Radius is a mean Proportional between AB and BC, is equal to all the conical Superficies, deſcribed by the revolution of ACEGB, on AB.

Cor. After the ſame manner it may be proved, that a Circle, whoſe Radius is a mean Proportional between CB & AN, is equal to all the conical Superficies deſcribed by ACEG on AN, in the Segment GAH.

THEOREM XV.

The Surface of a Sphere is quadruple of the greateſt Circle, made by a Section through its Center; i. e. to a Circle whoſe Radius is equal to the Diameter of the Sphere.

Let ADBK be a Circle, the Section of a Sphere through its Center.



Let there be inſcribed a regular Poligon, having an even number of Sides; draw the Diameter, AB, and the Diagonal BC, to the adjacent Angle.

Imagine the Circle ADBK, with the Poligon, revolved on AB; the Circle will generate a Sphere (NB. Def. 14.) and the Sides of the Poligon AC, CD, &c. will generate Right conical Surfaces, inſcribed in the Sphere.

AC and EB will deſcribe equal Cones (for they are equal, and equally inclined to the Axe AB); CD and DE will alſo deſcribe equal Surfaces; for they are equal, and CH is equal to EI.

The whole conical Superficies, is leſs than the Superficies of the Sphere; for the Arks, AFC, CGD, &c. are greater than the Chords AC, CD, &c; alſo, a mean Proportional, between AB and BC, is leſs than the Diameter, AB.

Let the Arks, AC, CD, &c. be biſected, at F and G; and draw AF, FC, &c.

It is manifeſt, that the conical Surfaces, deſcribed by the revolution of AFCGD, will be greater than thoſe deſcribed by AC, CD; but they are leſs than the Surface of a Hemiſphere, deſcribed by the Ark ACD; and, a mean Proportional between AB and BF, is greater than a Mean, between AB and BC, but it is alſo leſs than the Diameter AB.

By the ſame reaſoning, i. e. by multiplying the Sides of the inſcribed Poligon, the Superficies of Cones, inſcribed in the Sphere, will, at laſt, end in the Surface of the Sphere; when the Sides, of the Poligon end in the Circumference of the Circle. For, if AF be biſected, at O, the Chords AO, OF diviate but very little from the Arks they ſubtend; and a mean Proportional between AB and BO, will differ very little from the Diameter AB; and conſequently, it will at laſt be the ſame as the Diameter.

DEM. But, a Circle, whoſe Radius is equal to a Mean, between AB and BF, or between AB and BO, is equal to all the conical Superficies, deſcribed by the revolution of the Poligon, whoſe Sides are equal to AF or AO-Th. 14.

Conſequently, a Circle whoſe Radius is equal to the Diameter, AB, is equal to the Surface of a Sphere, deſcribed by the revolution of the Semicircle ADB.

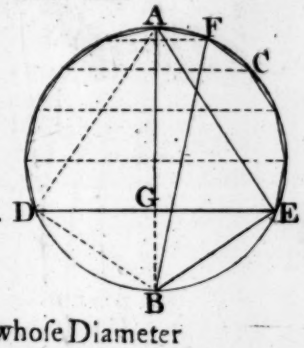
But, Circles are to each other, or amongſt themſelves, as Squares of their Diameters, - - - Cor. 1. 14. 6. and the Square of any Right Line is equal to four times the Square of half that Line, - - Cor. to 4. 2. Conſ. a Circle whoſe Radius is equal to the Diameter of another, is quadruple, i. e. equal to four times that Circle.

Th. the Superficies of a Sphere is equal to four great Circles, equal, in Diameter, to the Diameter of the Sphere.

COR. The Superficies of the Segment of a Sphere is equal to four Circles, whoſe Diameter is a Right Line drawn from the middle Point in its Surface (i. e. the Point where a Perpendicular, from the Center of its Baſe, cuts it) to the Circumference of its Baſe.

For, by the preceding, it is proved, that the conical Superficies, in every Segment, is equal to a Circle, whoſe Radius is equal to a mean Proportional, between the Chords CB, FB, or OB and AI, the Altitude of the Segment EAL.

Let DAE be a Segment, and AG its Axis. Draw AE.



Then, becauſe all the conical Surfaces, deſcribed by AF, FC, &c. is equal to a Circle, whoſe Radius is a mean Proportional between AG and BF; conſ. by multiplying the Sides of the Poligon, BF will coincide with AB.

But, AE is a Mean Proportional between AB and AG. - - - Cor. 2. 7. 6. Conſequently, a Circle, whoſe Radius is AE, is equal to the ſpherical Surface, deſcribed by the revolution of DAE.

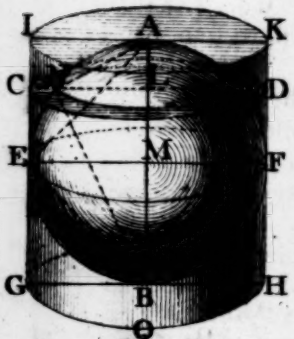
COR. 2. The Superficies of a Right Cylinder, whoſe Diameter and Altitude are equal, each to the Diameter of a Sphere, is equal to the Superficies of the Sphere.

For, the cylindrical Surface, being equal to a Circle whoſe Radius is a mean Proportional between its Side and Diameter, which are in this Caſe equal (Theo. 10.) is equal to four times its Baſe, i. e. to a great Circle of the Sphere.

But, the Surface of the Sphere is alſo equal to four ſuch Circles. Therefore, the cylindrical Surface is equal to the ſpherical-Ax. 3. 1

THEOREM XVI.

If a Sphere be inſcribed in a Cylinder, and being both cut by Planes, parallel to the Baſes of the Cylinder; each Segment of the ſpherical Surface will be equal to the portion of the cylindrical, intercepted between the ſame parallel Planes.



Let the Sphere, AEBF, be circumſcribed by the Cylinder GIKH, and let them be both cut by the Planes CD, EF, parallel to the Baſes, GH, IK, of the Cylinder.

I ſay, the ſpherical Surface NAO, or ENAOF, is equal to the cylindrical Surface ICDK, or IEFK; alſo, the ſpherical Surface NEFO, is equal to the cyl. Surface, CEFD.

Let AB, be the Axe of the Cylinder, and Sphere; and let the Diameters CD, EF, of the Cylinder, be drawn, cuting the Axe perpendicularly, at L and M, and the Surface of the Sphere at N and O, E and F. Draw AN, NB, and AE, EB.

DEM. Then becauſe ANB is a Right Angle (12. 3.) AN is a mean Proportional between AL and AB; i. e. between CI and CD. - - - - - - C. 2. 7. 6.

Alſo, becauſe AEB is a Right Angle, AE is a Mean, between AM and AB, i. e. between EI and EF. - ſame.

But, the Circle, whoſe Radius (AN) is a mean Proportional, between CI and CD, is equal to the cylindrical Superficies, ICDK. (10.) And, it is equal to the Superficies of the ſpherical Segment NAO. Therefore, the Superficies of the two Segments, of the Cylinder and Sphere, are equal.

After the ſame manner it may be proved, that the Superficies IEFK, of the Cylinder, is equal to the Surface of the Segment, ENAOF.

Conſ. the Surface of the Cylinder, CEFD, between the parallel Circles, CD and EF, is equal to the Segment of the ſpherical Surface, NEFO, between the ſame parallel Planes.

And, it is alſo manifeſt, that the remaining Superficies, EGQHF, of the Cylinder, is equal to EBF of the Sphere.

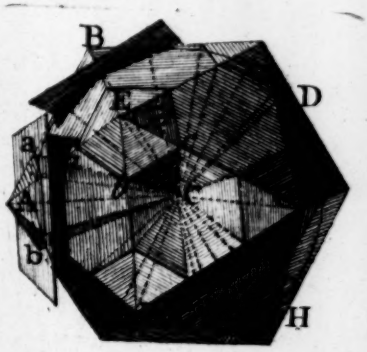
From this and the foregoing moſt excellent Theorems, is deduced a certain Rule for aſcertaining the true Area of the Surface of any Segment of a Sphere or other Portion, contained between parallel Circles.

Cor. The Segments of a ſpherical Surface, intercepted between parallel Circles, have the ſame Ratio, between themſelves, as the Segments of the Axe, or Diameter (AB) cut by theſe Planes.

For, each Segment, or portion of the ſpherical Surface, NAO, NEFO, and EBF, is reſpectively equal to the correſponding portion of the Cylinder, ID, CF, and EH; and thoſe Portions are, to each other, as IC to CE, to EG, i. e. as AL:LM:MB; conſequently, the Surfaces, NAO, NEFO, and EBF, are to each other, as AL, LM, and MB.

THEOREM XVII.

A Sphere is equal, in its ſolid Contents, to a Cone, whoſe Baſe is equal to the Surface of the Sphere, and its Altitude to the Radius.

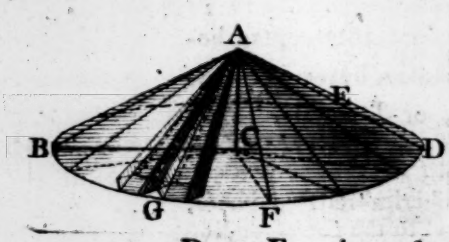


Imagine the Sphere ADF circumſcribed by any regular Body whatever; and, from every Angle of the circumſcribing ſolid ſuppoſe Right Lines, AC, EC, &c. drawn to the Center, C; there will be formed as many Pyramids, as the Solid has Faces, whoſe common Vertex is at the Center of the Sphere; which, together, is greater than the Sphere.

Imagine further, that every Angle, A, B, E, &c. of this Solid, be cut off by a Plane, touching the Sphere, and perpendicular to the Right Line from the Angle cut off. It is evident, that there will be formed another regular ſolid, containing the Sphere, touching every Face; which Solid is leſs than the firſt, but it is greater than the Sphere; and, by drawing Lines from every Angle, as before, there will be formed as many Pyramids as this Solid has Faces.

By proceeding thus, *ad infinitum*, it is manifeſt that the exterior Surfaces of this Solid, which conſtitute all the Baſes of the innumerable Pyramids, muſt, at laſt, end in the Surface of the Sphere; and conſequently, the Pyramids, or whole Solid, is the ſame as the Sphere; or, the difference leſs than any given Quantity.

Then, let BEF be a Circle, and C its Center. If the Radius BC be equal to the Diameter of the Sphere, the Circle BEF is equal to the Surface of the Sphere - Th. 15.



Make AC (perpendicular) equal to its Radius, and imagine the Right angled Triangle ABC revolved around; the Hypothenuſe, AB, will deſcribe the Surface of a Cone, BADF.

I ſay, the Cone is equal to the Sphere.

DEM. For, becauſe the Baſe of the Cone is equal to the Surface of the Sphere, it may be divided into as many plane Figures, as the circumſcribing Solid has Faces; each equal, reſpectively, to the other.

Conſequently, if Right Lines be drawn from every Angle of thoſe Figures, to the Vertex (as at G) there will be formed as many Pyramids, as the Solid has Faces.

But, they are, reſpectively, equal to them, by Hypotheſis; and they have, alſo, equal Altitudes; i. e. the Radius of the Sphere. Conſ. each Pyramid is equal to its correſponding one in the Solid; and, conſ. all the Pyramids contained in the one are equal to the other. But, Cones and Pyramids, whoſe Baſes are equal to one another, are equal. Th. the Cone, BADF, is equal to the Sphere ADF.

COR. 1. Sectors of Spheres, * are equal to Cones, whoſe Baſes are equal to the ſpherical Superficies of the Sectors, and the Altitude of the Cone to the Radius of the Sphere.

For, ſince the whole Sphere is equal to a Cone, whoſe Baſe is equal to the whole Surface, and Altitude to the Radius; conſ. a Cone having the ſame Altitude will be to the other, as the Baſe of one to the Baſe of the other; i. e. as the portion of the ſpherical Surface to the whole; and, conſ. the Sector is equal to ſuch a Cone.

COR. 2. A Hemiſphere is double of a Cone, having the ſame Baſe and Altitude.

For, becauſe the Surface of a Sphere is equal to four large Circles; the Surface of the Hemiſphere is double its Baſe.

But, a Cone whoſe Baſe is equal to the ſpherical Superficies of the Hemiſphere, and equal Altitude, is equal to the Hemiſphere; conſequently, if the Baſe of the Cone be equal to the Baſe of the Hemiſphere, it will be half the Hemiſphere.

COR. 3. A Segment of a Sphere is equal to the Difference between a Sector, whoſe ſpherical Superficies is the Surface of the Segment, and a Right Cone, whoſe Baſe is equal to the Baſe of the Segment, & its Side, to the Radius of the Sphere.

* By Sector of a Sphere is to be underſtood, a Portion containing a Segment, and a Right Cone on the Baſe of the Segment, whoſe Side is the Radius of the Sphere.

AN

APPENDIX;

ON THE THEORY

OF MENSURATION.

MENSURATION OF SUPERFICIES.

THE whole Theory of Menſuration of Superficies, conſiſts in finding a Rectangle equal to the given Figure.

In Theo. 17, Book 1ſt. it is demonſtrated, that every Triangle is equal to half a Parallelogram, on the ſame Baſe and Altitude; and, in Theo. 18th. is ſhewn and demonſtrated, that, all Parallelograms or Triangles, having the ſame or equal Baſes and the ſame Altitude, are equal. Conſequently, ſince Rectangles are Parallelograms (Def. 33 and 44) and it is, there, fully demonſtrated that they are all equal, having equal Baſes (i. e. having any two Sides equal) and the perpendicular diſtance between thoſe Sides and the oppoſite alſo equal, it is evident, that the meaſure of one is alſo the meaſure of the other. And, becauſe Triangles on equal Baſes with Parallelograms, and being of equal height are equal half ſuch Parallelograms, the Area of a Triangle is readily obtained.

Hence is deduced the general Rule (well known to all Surveyors or Artificers, concerned in meaſuring ſuperficies) for finding the Area of any triangular plane Figure; which is, to multiply the Baſe, i. e. any one Side, by half its height, from that Side, or half its Baſe by the whole height; the reaſon for which is, in thoſe Theorems, clearly accounted for; and, in that conſiſts the whole of ſuperficial Menſuration.

In Prob. 20th is ſhewn how to conſtruct a Rectangle, equal to a Triangle, on thoſe Principles; and in the 22nd it is further extended to a Trapezium; or to any Quadrilateral, whatever, regular or irregular.

The 21ſt ſhews how to conſtruct a Parallelogram, under any Angle, equal to a Rectangle, on the ſame Baſe; and conſequently, by changing the Premiſes, a Rectangle may be formed equal to any Parallelogram whatever.

If the conſtruction of thoſe Figures be well underſtood; practical menſuration is eaſily attained; which conſiſts (as I have obſerved above) in finding a Rectangle, i. e. in knowing how to take the dimenſions of the two Sides of a Rectangle, equal to any given Figure; for, the multiplication of any two Numbers being applied to meaſure, always denotes, or produces the Area of, a Rectangle under ſuch dimenſions; which I ſhall, in the firſt place, endeavour to make clear and intelligible.

Moſt People, at the firſt thought on theſe matters, imagine, that if two Figures (of any ſpecies whatever) have equal Circuit, i. e. if the meaſure of all the ſides, of each Figure, in one ſum, be the ſame, they have equal Areas; than which, nothing is more falſe, as will be made appear.

A Circle contains the greateſt Area of any Plane Figure, having an equal Circumference or Perimeter. If a Circle be depreſſed, though ever ſo little, it becomes elliptical; in which Caſe, it loſes of its dimenſions; and the more it is depreſſed, i. e. the more excentric an Ellipſis, having an equal Periphery, or Circumference with a Circle, the leſs is its Area to that of the Circle. Conſequently, if a cylindrical Veſſel be bulged, or if its Sides are depreſſed till it becomes elliptical, it loſes of its meaſure; and the more the Sides are depreſſed, the more it loſes; becauſe it is evident, that, if they are preſſed quite flat, it loſes the whole internal Area.

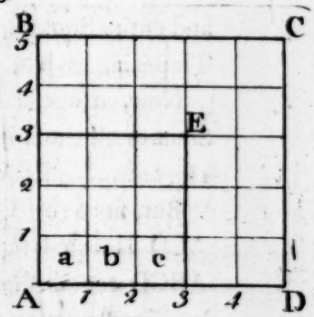
So likewiſe, of right lined Figures; the more the Sides are multiplied, and the nearer it approaches, in figure, to a Circle, the greater area it contains. Conſequently, regular Poligons, of a greater number of Sides, contain a greater Area, than thoſe of fewer Sides and having equal Perimeters.

Hence

Hence it is plain, that a Rectangle contains a greater Area than any other Quadrilateral whatever; if the meaſure of their Sides, in one Sum, be equal. But, a Square, which is the moſt perfect Rectangle, contains a leſs Area than a regular Pentagon; and a Pentagon contains leſs than a Hexagon; a Hexagon leſs than a Heptagon, and ſo on to a Circle, whoſe Perimeters are all equal; ſo that the Square and Circle, whoſe Perimeter and Circumference are equal, differ greatly in Area, as ſhall be illuſtrated hereafter to demonſtration.

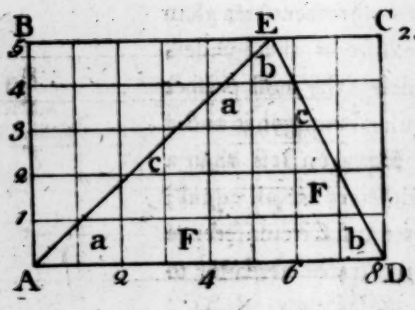
Now ſince, in Menſuration, a Rectangle is the ſtandard or criterion, by which the Area of all Plane Figures, as well as all other ſuperficial Contents, are aſcertained; I ſhall, in the firſt place, ſhew, and account for, the methods of taking the dimenſions of various Figures; which meaſures, being multiplied into each other, will give the true Area; each Figure being equal to a Rectangle under thoſe Dimenſions.

1. The area of a Square is obtained, by multiplying the meaſure of one Side into itſelf. e. g.



Let the Side of the Square ABCD be five feet; the meaſure of a Foot being repreſented by each ſmall diviſion on its Side, as 1, 2, 3, 4, 5. Now, if from theſe diviſions, lines are drawn, both ways, parallel to the Sides; it will be divided into as many leſs Squares, as the Side multiplied by itſelf produces; viz. 5 times 5, equal 25; which, is the number of ſmall Squares contained in the large Square, ABCD; each ſmall ſquare a, b, c, &c. being ſuppoſed one Foot, on each Side, is, therefore, called a ſquare Foot. So, the Square AE, containing three feet on each Side, is a Square Yard; its Area is nine ſquare Feet.

But, when a Rectangle meaſures more, or is longer, on one Side than the other; then, either Side multiplied by the other, not oppoſite, gives its Area or ſuperficial Meaſure. Whereas, in a Square, the Sides are all equal; and conſequently, the meaſure of one is alſo the meaſure of any other; therefore, the Side is ſaid to be multiplied into itſelf, to produce its Area.



2. If the Rectangle ABCD meaſures 8 Feet on one Side (AD) and 5 Feet on the other (AB) the oppoſite Sides are the ſame (15. 1.) and if Lines are drawn through the diviſions (as in the Square) parallel to the Sides, it will be divided into 5 times 8, equal 40; the number of ſquare Feet, contained in the Rectangle.

Hence, it is evident, that, in Menſuration, any two Numbers denoting Meaſure, in Inches, Feet, Yards, &c. being multiplied into each other, gives the area of a Rectangle under theſe Dimenſions, in ſquare Inches, Feet, &c. and conſequently, all multiplication of Meaſure, denotes the Figure, under ſuch dimenſions, to be right angled. Wherefore, the multiplication of Lines, in Geometry, implies a Rectangle conſtructed on two Lines.

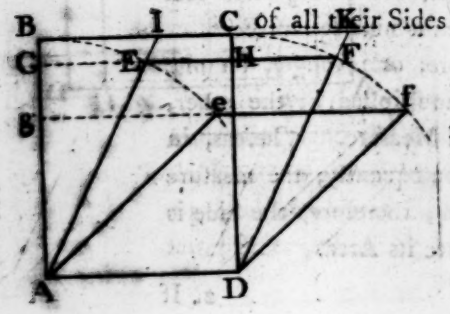
3. To any Point, E (in BC) let AE, DE be drawn, forming a Triangle, AED.

It is obvious, that the Triangle AED contains ſeveral whole and entire Squares, F, F, &c. and ſome Pentagons, as a, a; ſome Trapezia, as b b, and ſome Triangles, as c c, &c.

Now, it would be no eaſy matter to aſcertain how many entire Squares all thoſe irregular Figures are equal to; for there are but 12, entire, 4 Pentagons, 3 Trapezia, and 7 Triangles.

But, ſince (by Th. 17) we have full conviction, that the Triangle AED is equal to half the Rectangle ABCD; and the Rectangle ABCD contains 40 ſmall Squares; conſequently, all thoſe irregular Figures are equal to 8 Squares; which, added to the 12 entire ones, make 20, the true Area of the Triangle AED.

4. Next, I will ſhew, that two Parallelograms, may have the Sum of all their Sides equal, and differ greatly in Area.



The Rect. ABCD, and Par. AEFD, have their Sides, AD, BC, and EF, equal; for, AD is common to both; conſequently, BC and EF, being both equal to AD (15. 1.) are equal to each other - Ax. 3. AB=AE, and DC=DF (N. B. 1, Def. 20.) and conſ. AB+BC+CD+AD is eqnal to AE+EF+FD+AD.

But, the Area of the Par. AEFD is not equal to the Area of the Rect. ABCD. For, if AE be multiplied into AD, it will produce an Area equal to the Rect. ABCD, becauſe, AB=AE.

But, the Par. AEFD is equal to the Rect. AGHD only. For, it is demonſtrable, that, the Triangle AGE=DHF. - 7. 1. Conſequently, if DHF be taken away, and its equal, AGE, be added, the Rectangle AGHD is equal to AEFD.

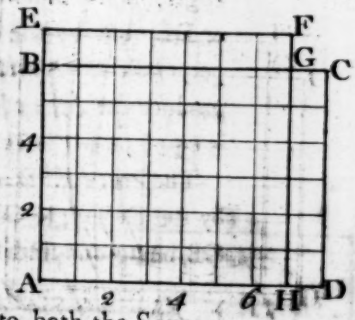
Again, if AE be produced to I, and DF to K (in BC produced) Then, the Parallelogram AIKD is equal to the Rect. ABCD. For, the Triangle ABI=DCK; wherefore, taking away DCK, and adding an equal, ABI, the thing is manifeſt.

Now, if the Rectangle ABCD be ſuppoſed to be depreſſed, to AEFD, it is evident that it has loſt of its dimenſions, conſiderably; and if it be depreſſed lower, to e f, it ſtill loſes more; notwithſtanding the Perimeter, A e f D, remains the ſame, equal ABCD. Conſequently, ABCD, contains a greater Area than any other Parallelogram, having an equal Baſe, and equal Perimeter.

Therefore, if ABCD be depreſſed, i. e. if the Angles are not Right ones, it will contain a leſs Area; for, if AB and CD deviate ever ſo little from a Perpendicular, BC muſt neceſſarily fall lower. Conſequently, its Altitude being leſs, and the Baſe remaining the ſame, its Area is leſs. - - - Cor. 1. 6.

5. It may ſeem ſtrange, to ſome, that a Square ſhould contain a greater Area than any other Rectangle, having equal Perimeters.

Suppoſe the Rectangle ABCD to meaſure, on one Side (AD) 8 feet, and on the other Side (AB) 6 feet; its Area is $8 \times 6 = 48$; and its Perimeter is $8+8+6+6=28$.



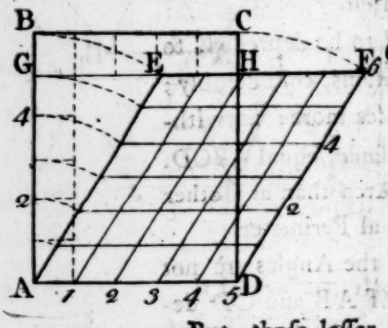
Let a Square, AEFH, be deſcribed on AH, equal 7 feet. Then, the meaſure of its Sides, in one ſum, is 7×4, or 4 times $7=28$, the ſame as the Rectangle.

Now, the Rectangle ABGH is common to both the Square, AEFH, and the Rectangle ABCD.

But the Rectangle, BEFG, which is the remaining part of the Square, contains 7 ſmall ſquares; whereas, the Rectangle DCGH, the remaining part of the Rectangle, ABCD, contains but ſix.

The area of the Square is 7×7; or 7 times $7 = 49$, whereas the area of the Rectangle is but $8 \times 6 = 48$; one 49th part leſs, in its Area, than the Square; which is accounted for by the Figure; and from the 11th Theorem, Book 5th.

Hence, it is plain, that a Square is the moſt perfect of Rectangles, or other Parallelograms; as a Circle, of all Plane Figures. And, ſince no other Figure, but Rectangles, can anſwer the purpoſes of Menſuration, in producing the true Area; it is certainly moſt conſiſtent, that all ſuperficial Meaſure ſhould be reduced to a ſtandard meaſure by the moſt perfect of Rectangles, a SQUARE.

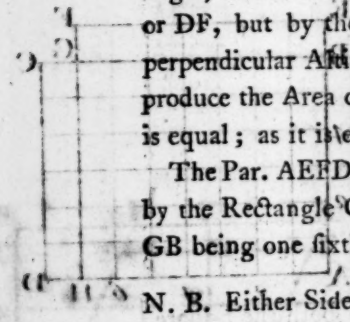


6. In the Parallelogram AEFD, let AE be divided into the ſame equal parts as AB or AD; and if, through thoſe diviſions, Lines are drawn parallel to the Sides, AD and AE, it will be divided into as many leſſer Parallelograms as the Rectangle, ABCD, contains Squares.

But, thoſe leſſer Parallelograms are Rhombuſes (Def. 36) each being in the ſame Ratio to a Square, of an equal Perimeter, as the whole Parallelogram AEFD to the Rectangle ABCD; conſequently, they do not aſcertain its Area.

The Area, therefore, of a Parallelogram (except it be a Rectangle) is not obtained by the meaſure of its two Sides, AD, AE, or DF, but by the meaſure of one Side, as AD, only, and its perpendicular Altitude, DH; which, being multiplied together, produce the Area of a Rect. AGHD; to which, the Par. AEFD, is equal; as it is evident from the Figure.

The Par. AEFD = AGHD, and is deficient of the Rect. ABCD, by the Rectangle GBCH, equal one ſixth part of ABCD (1. 6.) GB being one ſixth part of AB.



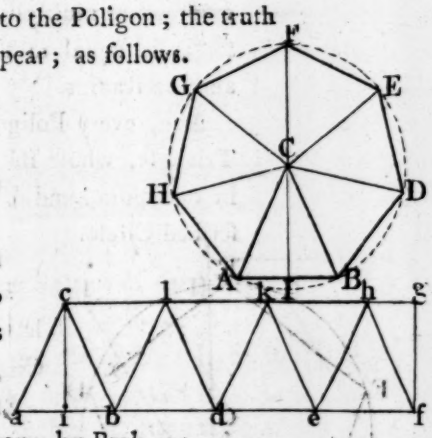
N. B. Either Side of a Parallelogram may be taken, in order to obtain its Area, but not both; unleſs it be a Rectangle.

If DF be taken, then, the perpendicular diſtance between that Side and its oppoſite, AE, will alſo give the true Area.

7. The

7. The Area of regular Poligons is readily obtained.

The general Rule for meaſuring Poligons is, to multiply half the Perimeter, i. e. half the Sum of all the Sides, by a Perpendicular from the Center to any Side; which implies, that a Rectangle of thoſe Dimenſions is equal to the Poligon; the truth of which I ſhall endeavour to make appear; as follows.



Let ABDEFGH be a regular Heptagon, whoſe Center is C.

Draw AC, BC, &c. dividing it into ſeven equal, Iſoſceles Triangles.

Draw a Right Line, a f, indefinite; in which, take a b, b d, &c. each equal to a Side of the Poligon; and, on the Baſes a b, b d, &c. conſtruct the Triangles, a c b, b l d, d k e, and e h f, equal and ſimilar to the Triangle ACB in the Poligon; by Prob. 12.

And, if a Right Line, c h, be drawn through their Vertices, it will be parallel to a f (C. 18. 1.) and, there will be conſtructed three more Triangles c b l, &c. equal and ſimilar to the other; and conſequently, the Trapezium, a c h f, is equal to the Heptagon AGEB: for, it contains ſeven Triangles, equal to thoſe in the Poligon.

Draw c i perpendicular to a b; the Triangle a c b being Iſoſceles, the Baſe, a b, and conſequently, the Triangle, a c b, is biſected, by the Perpendicular c i. - - - C. 3. 9. 1.

Produce c h; and, make h g equal a i and draw f g, which will be parallel to c i, and alſo equal. - - - 15. 1.

wherefore, the Tri. f g h, is equal to the Tri. a c i. - 7. 1.

conſequently, the Rectangle i c g f is equal to the Trap. a c h f; (which was made equal to the Poligon.) - - Ax. 6 and 7.

But, the Side i f, equal c g, of the Rect. i c g f, is equal to IBDEF, half the Perimeter of the Poligon; and c i, equal g f, the other Side of the Rectangle, is equal to CI, a Perpendicular of the Poligon; by Conſtruction.

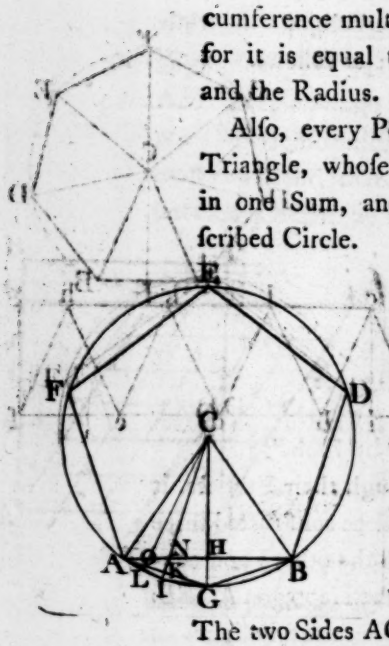
Conſequently, half the Sum of the Sides, AB, BD, &c. (equal i f) multiplied by the Perpendicular CI (equal c i) gives the Area of the Heptagon ABDEFGH.

Which, Rule, holds good for all regular Poligons whatever.

8. From

8. From what has been advanced, concerning Poligons, we do with certainty infer, that a Triangle whoſe Baſe is equal to the Circumference of a Circle, and its Altitude equal to the Radius, is equal to that Circle; and conſequently, half the Circumference multiplied by the Radius gives the Area of a Circle; for it is equal to a Rectangle, under the ſemi-Circumference and the Radius.

Alſo, every Poligon, circumſcribing a Circle, is equal to a Triangle, whoſe Baſe is equal to the meaſure of all the Sides, in one Sum, and its Altitude equal to the Radius of the inſcribed Circle.



The Pentagon ABDEF, it is evident, is leſs, in its Area, than the Circumſcribing Circle; by the five Segments AGB, BD, &c. Biſect the Ark AGB and draw CG, which will be perpendicular to AB; and draw the Chords AG, GB, which are the Sides of a Decagon, inſcribed in the ſame Circle; and, if the Side AG be biſected, and CK drawn, it will be perpendicular to AG. - Cor. 3. 9. 1.

The two Sides AG, GB, of the Decagon, are greater than the Side AB, of the Pentagon (13. 1.) and, the Perpendicular CK, of the Decagon, is greater than the Perpendicular CH, of the Pentagon; for, CN is longer than CH - - - C. 2. 12. 1.

And CK is longer than CN; conſequently, the Area of the Decagon, whoſe Perimeter is ten times AG or GB, and its Perpendicular is CK, muſt be greater than the Pentagon, whoſe Perimeter is five times AB, and its Perpendicular is CH.

The difference is five Iſoſceles Triangles, equal ABG; whoſe Baſe is AB, the Side of the Pentagon, and the equal Legs or Sides, AG, and GB, are the Sides of the Decagon.

Again, the Area of the Decagon is leſs than the Circle, by the ten Segments AIG, GB, &c.

CK, produced, biſects the Ark AIG at I; draw the Chords AI and IG; which are the Sides of a Poligon of twenty Sides, whoſe Perpendicular is CO.

Th is

This Poligon is greater than the Decagon, by ten Iſoſceles Triangles, equal AGK, in the Segment AG or GB; and it is ſtill leſs than the Circle by twenty Segments AI or IG.

9. From all which, it is clear, that every Poligon, inſcribed in a Circle, is leſs than the Circle, by as many Segments as the Poligon has Sides; and conſequently, every Poligon, circumſcribed, is greater than the Circle.

But, the greater the number of Sides of the Poligon, the nearer it is to the Area of the Circle, i. e. to the Circle itſelf; and conſequently, it muſt at laſt end in the Circle; that is, the Perimeter of the Poligon, will be equal to the Circumference of the Circle, and the Perpendicular (from the Center) to the Radius.

For, if AI be again biſected in L; the Sides AL, LI, of a Triangle, in the Segment AI, are greater than the Chord AI, which is the Baſe of that Triangle; yet, they are leſs than the Arks of the Segments AL, LI; but they muſt, at laſt, by multiplying them, end in the Circumference.

So likewiſe, the Perpendicular CH, of the Pentagon, is leſs than CK, the Perpendicular of the Decagon, which is alſo leſs than the Perpendicular CO, of the Vigintagon. And, if AL is again biſected, and a Perpendicular drawn to the Chord AL, it will be greater than the Perpendicular CO; and will at laſt be equal to the Radius of the Circle.

Wherefore, the Circle being conceived to be formed of an infinite number of Iſoſceles Triangles, whoſe common Vertex is the Center, and whoſe Baſes are in the Circumference, and which are, altogether, equal to it; conſequently, a Rectangle under the Semi-circumference and the radius of the Circle; or, under half the Perimeter of the innumerable ſided Poligon (equal to the Circumference) and its Perpendicular (equal the Radius) is equal to the Circle.

And conſequently, a Triangle whoſe Baſe is equal to the Circumference, and its Altitude equal to the Radius, is alſo equal to the Circle; for every Triangle is equal to half a Rectangle of the ſame Baſe and Altitude (17. 1.) Therefore the Triangle, is

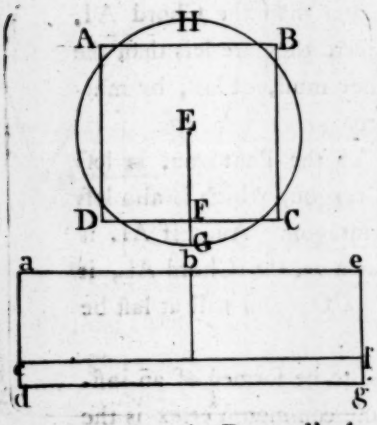
equal to a Rectangle under half its Baſe, and the Altitude. Hence, the Rule for meaſuring a Circle is, to multiply half the Circumference by the Radius.

From hence it is clear, that, of all Figures which have an equal Perimeter or Circumference; i. e. whoſe Bounds are equal to the ſame Right Line; a Circle contains the greateſt Area.

10. I have already ſhewn, that a Square contains a greater Area than any other Rectangle, having an equal Perimeter, or the meaſure of all their Sides equal. I will next ſhew, how much it is leſs than a Circle, having equal circuit.

Suppoſe the Sides of the Square, ABCD, be each equal to a fourth part of the Circumference of the Circle, GH; then the four Sides are equal to the whole Circumference.

The Area of the Circle is greater than the Area of the Square.



From what has been ſaid, concerning Poligons and Circles, it is evident, that the Circle, GH, is equal to a Rectangle, two oppoſite Sides of which, are equal to the Circumference; the other are each equal to the Radius EG.

But, the Square is equal to a Rectangle, two oppoſite Sides being equal, each to two ſides of the Square, i. e. equal half the circumference of the Circle; and the other two Sides, each equal to the Perpendicular, EF, only; which, being conſiderably leſs than the Radius EG; conſequently, a Rectangle, a e g d, whoſe Sides a e, d g, are each equal half the Circumference, GH (equal a e, c f, of the Rectangle a e f c) and the Side e g (equal a d) is equal the Radius, EG (but, the Sides a c, e f, each equal the Perpendicular EF, only) will have a greater Area than the Rectangle a e f c; and the difference is the Rectangle c f g d.

The Circumference of a Circle is to its Diameter, nearly, as 22 is to 7; which, is near enough, for meaſuring any Circle that can be formed. But more exact, the Diameter being 113, the Circumference will be 355. In both, the Circumference is too much; the laſt, by little more than one five millionth part of the Diameter.

Suppoſe then, by the laſt, the Diameter, GH, to be 17, the Circumference will be 53,4, very near. The Side of the Square, ABCD, being a fourth part of the Circumference, is 13,35, the Perpendicular, EF, half the Side, is 6,675; and the Radius of the Circle EG, is 8,5; the Area of each is as follows.

AB + BC = 26,7	half the Circumference 26,7
× by EF = 6,675	× by the Radius, EG, = 8,5
1335	1335
1869	2136
1602	
1602	Area of the 226,95 Circle
Area 178,2225 of the Square	178,2225
	difference 48,7275

A regular Pentagon, having the ſame Perimeter as the Square, is greater than the Square, but it is leſs than the Circle; alſo a Hexagon, is ſtill greater than a Pentagon, of equal Perimeter. A Heptagon, is ſtill greater than a Hexagon; and an Octagon, greater than a Heptagon; ſo that, the more the Sides are multiplied, the nearer it approaches to the Circle; for, the Perimeter of the Poligon, by multiplying, will end, or fall, at laſt, into the Circumference of the Circle.

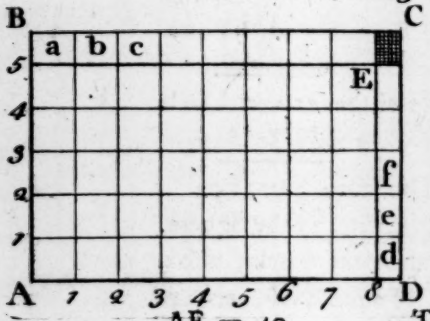
11. From what has been advanced, it is manifeſt that the whole theory of Menſuration, of Superficies, conſiſts in finding the ſides of a Rectangle, equal to the given Figure; ſince the Multiplication of any two meaſures (into each other, or one by the other) denotes a Rectangle of thoſe dimenſions.

Having ſhewn how the meaſures of Plane Figures, from a Square to a Circle, are taken, ſo as to produce their Areas; I ſhall, in the next place, account for the multiplication itſelf, by which, the true Area is aſcertained.

Suppoſe, then, a Rectangle ABCD, whoſe Sides, AB and CD, meaſure, by ſome Scale of equal Parts, 5 and 9 twelths, each; and the other Sides, AD and BC, each 8 and 6 twelths.

Through the Diviſions 1, 2, 3, 4, &c. on the Sides AB and AD, which may repreſent the meaſure of a Foot or 12 Inches, each (for, you may ſuppoſe any meaſure whatever, to be repreſented by them) if Lines be drawn, parallel to the Sides of the Rectangle, as in the Figure, it will be divided into as many whole

and entire Squares, as the whole numbers produce; viz. the Rectangle AE, whoſe Sides, A 8, A 5, being multiplied, 5 times 8, is equal to 40. The fractional parts a, b, c; d, e, f, &c. produce 8 more Integers or Squares, and 7 eighths of another; for, the 8 ſmall Rectangles a, b, c, &c. to E, are each 3 fourths of a Square; being the full meaſure on one Side, and but 9 twelths on the other; and are, altogether, equal to 6 entire Squares.



The 5 ſmaller Rectangles d, e, f, &c. to E, being but 6 twelths (half the Integer) on one ſide, are each equal to half a Square; and make, together, two and a half; and the ſmall Rectangle EC, being 3 fourths of one of theſe, or half one of the other, is equal 3 eighths of a Square.

AE =	40
BE =	6
DE =	2 - 6
CE =	0 - 4½
ABCD	48 - 10½

The large Rectangle, AE, is equal to 40 Squares; to which, if the Rectangles BE, DE, and EC be added; BE=6, DE=2½, and EC=⅜ of a Square; together, they are equal to 8, and 7 eighths; which, added to 40, makes 48 and 7 eighths; the ſum total of the Area of the Rectangle ABCD, in ſquare meaſure, by that Scale.

12. Next, I will ſhew how the Product ariſes, by multiplying Feet and Inches, together; commonly called Croſs Multiplication.

The meaſures of the two Sides of the Rectangle, AB and AD, 8 - 6, and 5 - 9, being placed one under the other ('tis not material which is firſt, or uppermoſt) as in the margin, we begin with the Feet, in the multiplier, and ſay, 5 times 6 is 30.

8 - 6	
5 - 9	
42 - 6	
6 - 4 - 6	
48 - 10 - 6	

Now, it is evident, from the Figure, that this is Feet, one way, and Inches, the other; and is the Rectangle DE, 5 feet long, E 8, by 6 Inches, 8 D.

Feet multiplied by Inches, or Inches multiplied by Feet, produce Inches; i. e. the twelfth part of a ſquare Foot; 30 Inches is, therefore, equal to 2 feet and 6 Inches.

'Tis the ſame, if you begin with the 9 inches, firſt.

Set

Set down the 6 Inches, under Inches, and carry the two Feet forward to the whole Numbers, and ſay, 5 times 8 is 40; which being Feet both ways, they conſequently produce ſquare Feet, and is the large Rectangle, AE. The two Feet, ariſing from the Rectangle BE, bing added, make 42 Feet; which, ſet down under the Feet.

Next, take the Inches, in the multiplier, and ſay, 9 times 6 is 54.

Now, theſe are Inches both ways (the ſmall Rectangle EC) and produce only Parts, or ſquare Inches, i. e. the 144th part of a ſquare Foot. For if the Rectangle EC be divided, on each Side, into Inches (viz. 9 and 6) and lines be drawn through them, as in the large Rectangle AE, it will contain 54 ſquare Inches, which are called Parts; twelve of which are called an Inch; for, it muſt be obſerved, that what is here meant by an Inch, is not a ſquare Inch, as by a Foot is underſtood a ſquare Foot; but it is the 12th part of a Square Foot, equal 12 ſquare Inches, or Parts.

Wherefore, the ſmall Rectangle EC, which contains 54 ſquare Inches, it is evident, is equal to 4 Inches and 6 Parts only; ſince twelve of theſe Parts are but one Inch, and 4 times 12 is 48, there remains 6 Parts, or half an Inch; which, ſet down, one place to the right hand of Inches, and carry forward the Inches to the next place.

Laſtly, ſay 9 times 8 is 72, which is the Rectangle BE, 8 Feet one way (5 E) by 9 Inches the other (5 B) and produces Inches; to which, add the 4 Inches from the laſt place, it makes 76 Inches, equal 6 Feet 4 Inches; ſet them down, as above, Feet under Feet, and Inches under Inches; which, added into one Sum, makes 48 Feet, 10 Inches, and 6 Parts (equal half an Inch) the Area of the whole Rectangle ABCD, in ſquare Feet, Inches, and Parts.

Thus, multiplying Feet and Inches together, the Area of a Rectangle is obtained, whoſe Sides are equal to ſuch Dimenſions. 'Tis needleſs to proceed further; as it is evident, if Parts were taken, they would be accounted for, after the ſame manner.

13. As there are many Perſons, who meaſure accurately by Feet and Inches, yet are unacquainted with Decimals; I ſhall, briefly, explain the manner of multiplying Decimals; which is much eaſier, and better adapted to any Quantity whatever.

Multiplication, by Feet and Inches, denotes the fraction of a Foot to be in twelfth parts, which are called Inches; and these are again divided into twelve, called parts, and so on to Seconds and Thirds; each of which is a twelfth part of the foregoing. Whereas, the Decimal Fraction implies the Integer, whatever it be, to be divided into tenths, hundredths, and thousandth Parts, and so on, by tens, *ad infinitum*, as the other by twelves, and are therefore called Duodecimals.

Multiplication, in Decimals, is performed in the same manner as whole Numbers; seeing, that the places increase regularly, by tens, from the Decimal to the Integer, as in whole Numbers.

Observe the following Example; by which, the Area of the Rectangle ABCD is produced, the same as by Feet and Inches.

5,75
8,5
2875
4600
48,875

The measures of the Sides of the Rectangle, in Decimals, are as in the margin. AB, 5 feet 9 Inches, is, in Decimals, 5 Feet and 75 hundredth parts of a Foot, equal 9 Inches; and AD, 8 Feet and 6 Inches, is 8 feet, and 5 tenths of a Foot, equal 6 twelfths. I place the least Side, AB, first, because it has more places of Figures, and would occasion another Line, if placed otherwise, the Product would be the same.

In the operation, the Figures are multiplied together, as in common Arithmetic, without any regard to the places of Decimals or Integers; and, when added together, as many places of Decimals as there are in both the Multiplicand and Multiplier, viz. three, are taken from the right hand of the Product; the remaining two are Integers, or Feet; which produces 48 Feet, and 875 thousandth parts of a Foot, equal 7 eighths of a Foot, equal 10 Inches and a half. For, 500 is half a Thousand, therefore equal to 6 Inches; and the remaining 375 is 3 eighth parts of a Thousand, therefore equal to 4 Inches and a half, which is 3 eighth parts of a Foot.

The value of a Decimal Fraction is easily obtained, as follows.

Let the Integer be whatever it may; multiply the Decimal by the number of parts of the next inferior denomination of the Integer; which, in this Case, is Inches, or the twelfth part of a square Foot, to which I would reduce the Decimal 875.

Multiply

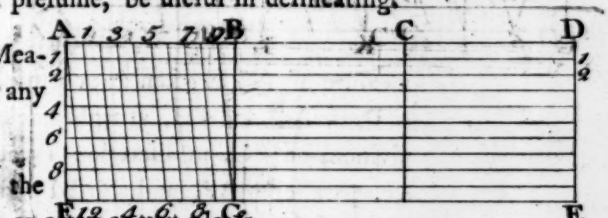
Multiply it by 12, and take away from the Product, as many places as you had at firſt, there is left 10 Inches; and the Decimal, 500, are now thouſandth parts of an Inch; conſequently, equal to half an Inch, or 6 Parts, as a ſecond operation evinces; 12 being, again, the next inferior Denomination, as twelve Parts are equal to one Inch.

875
12
10,500
12
6,000

If the original ſtandard meaſure of a Foot was, inſtead of Inches, divided into tenth parts of a Foot; or, which is the ſame thing, if an Inch had been made a tenth part of a Foot, inſtead of a twelfth part, and the Inches ſubdivided into tenths inſtead of eighth parts, Menſuration would have been greatly facilitated, by decimal Parts inſtead of Duodecimals, or multiplication of Feet and Inches. The meaſures being taken, at once, in Decimals (which, by a common Rule, is difficult to do, accurately, and loſs of time to reduce ſome meaſures to a Decimal) would be more uſeful in all Caſes whatever.

I am not a little ſurprized, that there has been no attempt to make it univerſal, as every Perſon, who has experienced it, muſt give the preference to Decimals, which is infinitely preferable to any other manner of dividing.

14. I ſhall, here, ſhew how to make a Scale, and divide it into Decimal Parts, which may, I preſume, be uſeful in delineating.



Let AB be a determined Meaſure repreſenting one Foot, or any other meaſure required.

Produce AB, and repeat the meaſure of AB, to C and D, as often as you pleaſe.

Draw AF and DE perpendicular to AD; on either of which, make, at pleaſure (i. e. with any opening of the Compaſſes) ten equal diviſions, as on AF; and, through them, draw lines parallel to AD, as 1, 1 and 2, 2, &c. in all, eleven lines.

Divide AB and FG into ten equal parts (Prob. 36) and draw A 1, 1 2, &c. diagonal-wiſe, as in the Figure, and proceed in the ſame manner, to 9 G, parallel to each other.

By

By which means, the length or meaſure given, AB, is divided into a hundred equal parts; each interval, from AF to A1, the firſt Diagonal, at 1, 2, 3, &c. expreſſing all the Units from 1 to 10.

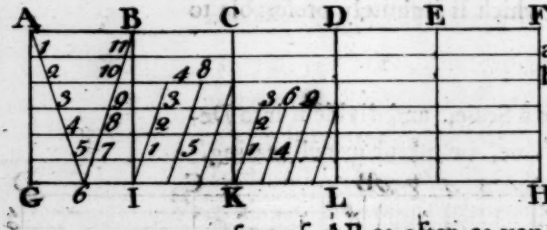
Between each of the Diagonals, as A1, and 1 2, &c. the intervals are ten of thoſe hundredth parts, being every where equal to the Diviſions on AB.

In the ſame manner, any Meaſure whatever, either greater or leſs than AB, may be accurately divided into Decimal Parts.

After ſuch plain directions, in the Conſtruction, the application of it, in Practice, would be impertinent and unneceſſary.

15. In the ſame manner, a Scale may be divided into Duodecimals; by taking 12 Diviſions on AF, and dividing the given Meaſure, AB, into twelve, and proceeding as directed above.

Duodecimals is ſeldom practiſed, further than the diviſion of a Foot into Inches, which are the ſtandard Meaſure, on all common Rules; therefore, to make a Scale of Feet and Inches, to any determinate meaſure, obſerve the following directions.



Let AB be the determinate meaſure of a Foot, according to the proportion you intend to make a Drawing, or Deſign.

Produce AB, and repeat the meaſure of AB as often as you think neceſſary, as C, D, E, F.

Draw AG and FH, at right angles to AF, and make ſix equal diviſions on AG, at diſcretion; through which, draw 1 a and 2 b, &c. parallel to AF, and alſo BI and CK, &c. parallel to AG.

Biſect GI, or AB, and draw the Diagonals, A6 and B6; and the given Meaſure, AB, is divided into twelve equal parts, as they are numbered from the Perpendicular AG, on the Diagonals.

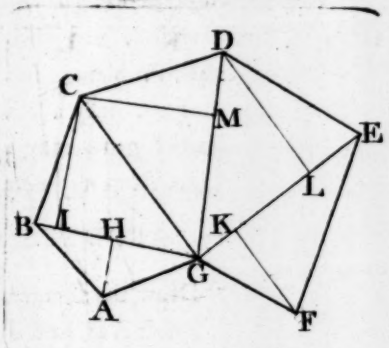
N. B. It may be done with only four, or three Spaces, as between IK and KL; but I think it more applicable, in Practice, with ſix; which is left to every Perſon's own diſcretion; the Diviſions are the ſame in them all.

Having

Having illuſtrated and explained the whole rationale of menſuration of plane Superficies, and ſhewn, how the Dimenſions of all regular Figures are taken, in order to find Rectangles equal to them; I ſhall juſt give one ſpecimen of irregular Figures; for, I do not intend this as a Treatiſe on Menſuration, but only a brief Theory, to ſhew how it is founded in Geometry; which, being well underſtood and digeſted, it is eaſy to be applied in all Caſes whatever, which may occur, in Practice.

Menſuration of Superficies, applied to various kinds of Artificers Work, is ſtill the ſame in Theory, but much eaſier, in Practice, than Surveying Land; on account of the irregularity of its Figure, and unevenеſs of its Surface. The former depends, chiefly, on the mode, or cuſtom of taking their dimenſions, in order to reduce the Work to ſquare meaſure; which is not ſo eaſy to deſcribe, but requires very little application under a Proficient. For, a Perſon ever ſo well verſed in Theory, being a ſtranger to practice, would differ in his eſtimate, for no other reaſon but taking his dimenſions different; in which they are not ſo very exact, but give and take all reaſonable allowances; inſomuch, that two Meaſurers, of long practice, taking their dimenſions, ſeparately, would differ conſiderably in their Sums total.

16. ABCDEFG is an irregular Heptagon; having 7 Sides.



Draw the Diagonals BG, CG, GD and GE, or otherwiſe, at diſcretion, dividing the given Figure into Triangles, as ABG, BCG, &c. in each Triangle, draw Perpendiculars, as AH, CI, &c. The meaſure of a Perpendicular and half the Baſe on which it falls, in Feet and Inches, or Decimal Parts, being multiplied into one another, gives the Area of each Triangle ſeparately; which, added into one Sum, is the Contents of the whole Figure (Ax. 2. 1.)

Or it may be done ſomewhat readier, by taking any two contiguous Triangles, as ABG, BCG, which together, make a Trapezium, ABCG; unleſs it ſhould happen to be a Parallelogram.

The Diagonal, BG, is then a common Baſe to the two Triangles and the Area, of both, is obtained at once, by multiplying the Diagonal, BG, by half the Sum of the two Perpendiculars (Prob. 22.)

The Area of any other two contiguous Triangles, as GDE, GEF, forming a Trapezium, is had after the ſame manner; and if another Triangle remains, as GCD, its Area, being obtained, muſt be added to the Areas of the two Trapeziums, into one Sum; which is the true Area of the Heptagon ACDF.

In this Proceſs it may be obſerved, that not one Side of the Figure is meaſured, but only the Diagonals and Perpendiculars, all which fall within the Figure. The reaſon is obvious; becauſe the Sides are not at right angles with each other; or if ſome of them were, it would be of no uſe, unleſs they formed a Rectangle, or other Parallelogram, with the adjacent Diagonal, which is ſeldom the Caſe. Whereas, the Perpendiculars making Right Angles with the Diagonals, it is eaſy to account for the true Area being obtained by them; from 17 and 18th of the 1ſt of Elements.

N. B. If the Figure to be meaſured be a large Field, the meaſures are taken in Chains and Links, &c.

As it muſt be obvious to every one, who has gone thus far, that to reduce any irregular Figure to a Triangle, having an equal Area to the Figure, is the principal buſineſs of Menſuration of Superficies; I would adviſe the learner to be clear in the Conſtruction, and alſo in the Demonſtrations, of the 20, 21, 22, and 24th Problems. In reſpect of the 26th and 27th, I ſhall add further; that not only a Triangle, or Trapezium, but almoſt any right lined Figure may be divided into two equal parts by a Right Line, from any determined Point, in any Side; as follows.

17. ABCDEF is a Hexagon, having an internal Angle, at D.

Draw the Diagonal AD, dividing it into two Trapeziums.

Biſect AD, and divide each Trapezium, AC and AE, into two equal parts, by the Right Lines GH and GI, (Prob. 27.)

The whole Figure is divided into equal parts; HAI = CGE.

Draw HI, and GB parallel to it; then, a Right Line, BI, divides the whole Figure equally.*

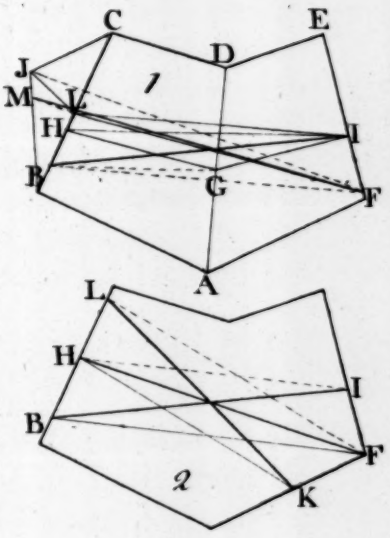
Now,

Now, B I biſects the hexagonal Figure A C F; but, the Points B and I are accidentally obtained.

Fig. 2. Let K be the given Point, from which a Right Line is required to be drawn, biſecting the given Figure.

Draw BF, and IH parallel to it, and draw FH; which alſo biſects the Figure.

Draw KH, and FL parallel to it, and join KL. I ſay, KL divides the whole Figure equally in two.



The proof is in the 18th of the 1ſt of Elements, the ſame as in the 27th Problem; for, the Triangle B H F = B I F, and H L K = H F K.

After the ſame manner it may be done from any Point whatever, in any Side.

If the Figure had more Angles, as B J C (Fig. 1.); a Right Line may be drawn, after the ſame manner as B I, dividing the whole Figure equally; having perhaps two or three more operations. For, by biſecting B C, and drawing J L, the Point, where B I cuts BC, may, by one more operation, be brought to L (as KL) and then by joining K J and drawing L M parallel, KM, will biſect the whole Figure.

18. Now, ſeeing that all irregular Figures are reduced, to Triangles, in order to obtain their Areas; I will ſhew how the Area of any right-lined Triangle may be obtained without having a Perpendicular, by the meaſures of its Sides only.

In ſurveying Land, it may ſo happen, that, on account of Bogs, Water, Trees, &c. it would be very difficult, if not impoſſible, to obtain a Perpendicular, within the Figure, when its Sides are acceſſible. In ſuch Caſe, the Perpendicular may be found arithmetically, deduced from the 13th of the 2nd Book; or, the Area may be had without the Perpendicular, by the following Rule.

* Let it be obſerved, that if GB had fallen on this Side of the Angle (which is very probable) it would then be neceſſary to have recourſe to the third operation (Fig. 3.) in the 27th Problem; by which, the Line, BI, biſecting the Figure, would be obtained on this Side of the Angle B.

From half the Sum of all the Sides of the Triangle, ſubtract each Side ſeverally; multiply the half Sum by the three Differences, continually; out of which Product, extract the Square Root; which is the Area of the Triangle, e. g.

Let the Sides of the Triangle ADB, be, AB–9, 8; AD–8, 5; and BD–6, 7. The half of which Sum is 12, 5; and the Differences, ſubtracting each Side, ſeverally, are, AB–2, 7; AD–4; and BD–5, 8. See the whole Operation, as follows.

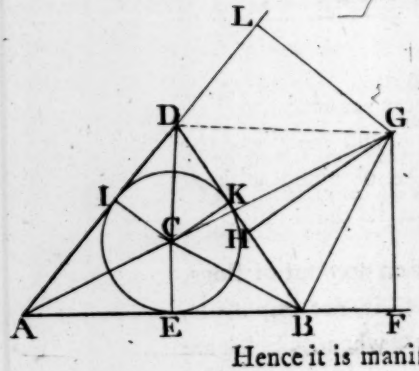
AB—9,8
AD—8,5
BD—6,7
25,0
12,5 half the Sum of the Sides.

Differences { 2,7 / 4,0 / 5,8 }

12,5
2,7
875
250
33,75
4,
135,
5,8
1080
675

783 (27,98
4 Area.
47) 383
329
549) 5400
4941
5588) 45900
44704
1196

The half Sum, 783, multiplied into the Differences.



In the Triangle ADB, biſect every Angle by Right Lines, meeting at C; which is the Center of a Circle inſcribed. - - 4. 4.

The Triangle ABD is thus divided into three Triangles ACB, BCD and ACD; the Perpendiculars of all which, CE, CI, and CK, are equal; and conſequently, AE, is equal to AI; BE = BK, and DI = DK - C. 2. 16. 3.

Hence it is manifeſt, that AE × CE is equal to the Trap. AECI. (For, CE and CI are, reſpectively, perpendicular to AB and AD) Alſo, BE × CE = Trapezium BECK, and DI × CI = DICK. Produce AB; make BF equal to DI, equal DK.

Conſ. AF × CE is equal to them all, i. e. to the Triangle ABD.

I ſay, the Square Root, of AF × AE × EB × BF, is alſo equal to the Triangle.

DEM.

DEM. Becauſe AE=AI; BE = BK; and BF equal DK - Con. AF is equal to half the ſum of the Sides, AB, BD, and AD. Alſo, AE, EB, and BF, are reſpectively, the Differences of thoſe Sides, ſubtracted from the half of their Sum; (for, AE=AF−BD; EB=AF−AD; and, BF=AF−AB.) Draw FG perpendicular to AF, produce AC to G, and draw GB. GB will biſect the external Angle, DBF, of the Triangle.

Draw GH perp. to BD; the Trap. FGHB is ſimilar to EBKC. For, the Angles at F and H, E and K are Right; by Conſtruction; wh. the Angle FBH+FGH= to two Right Angles - Th. 1. 10. 1. and, FBH+HBE are equal to two Right Angles - - - 1. 1. conſ. ECK=FBH (for, HBE+ECK=2 Right Angles) Ax. 7. 1.

But, BC biſects the Angle ECK, by Conſtruction; & CE=CK wherefore, BG biſects the Angle FBH; and BH is equal to BF; and, conſequently, the Triangles CBE, BGF are ſimilar - 13. 6. Alſo, becauſe FG is parallel to CE, the Triangles ACE, AGF are ſimilar; (2. 6.) and therefore, CE:AE::FG:AF - 4. 6. Alſo, becauſe BCE, BGF are ſimilar, CE:EB::BF:FG. and conſequently, CE × CE × AF × FG = AE × EB × BF × FG.

Now, becauſe FG is on both Sides, let it be rejected, and ſubſtitute AF, on both Sides; and conſequently, it will be CE × CE × AF × AF, i. e. CE □ × AF □ is equal to AE × EB × BF × AF; i. e. to AF, the half Sum of the Sides, multiplied into the three Differences AE, EB, and BF.

But, AF multiplied by CE is equal to the Area of the Triangle; conſ. it is the ſquare Root of CE □ × AF □; which is equal to AF, multiplied into the three Differences. Q. E. D.

Menſuration of curved Surfaces, convex or concave.

18. The Areas of the Superficies of regular curved Surfaces, are eaſily deduced from the 10th, 11th, 12th, the 15th and 16th Theorems, Book the 8th.

By the 10th, the curved Superficies of a Cylinder is equal to a Circle, whoſe Radius is a mean Proportional between its Side and Diameter.

Or, it is obvious, from the 5th, that it is equal to a Rectangle, whoſe Sides are its Circumference and Height.

19. By

19. By the 11th it appears, that the conical Superficies of a Right Cone, is equal to a Circle, whoſe Radius is a mean Proportional between its Side and the Radius of its Baſe.

Or, it is equal to a Triangle, whoſe Baſe is equal to the Circumference of the Baſe of the Cone, and its Altitude, to the Side of the Cone.

20. By the 12th it is manifeſt, that the conical Superficies, of the Fruſtum of any Right Cone, is equal to a Circle, whoſe Radius is a mean Proportional between the Side of the Fruſtum, and the Radius of the Baſe, added to the Radius of the oppoſite Circle.

And every Circle is equal to a Triangle, whoſe Baſe is equal to its Circumference, and its Altitude to the Radius. Art. 8th.

Or, the conical Superficies is equal to a Rectangle, whoſe Sides are equal to the Side of the Fruſtum, and half the Sum of the Circumference of the Baſe, added to half the Circumference of the oppoſite Circle.

21. The Area of the Surface of a Sphere is equal to a Circle, whoſe Radius is equal to the Diameter of the Sphere. - 15.8.

And the Area of any ſpherical Segment is equal to a Rectangle, one Side of which is equal to the Circumference of the Sphere, the other to the height of the Segment.

For, it is equal to a cylindrical Surface of thoſe Dimenſions - 16.8.

The ſame holds true of any portion of the Surface of a Sphere, intercepted between parallel Planes.

The ſpherical Surface, intercepted between parallel Planes, is therefore equal to a Rectangle, whoſe Sides are the Circumference of a large Circle of the Sphere, and the perpendicular diſtance between the parallel Circles.

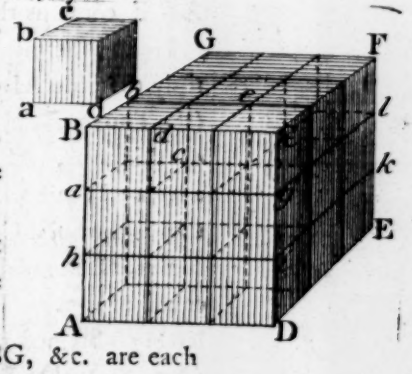
Theſe Rules, reſpecting curved Surfaces, being clearly underſtood, will be found extremely uſeful to Artificers, in meaſuring circular Halls, or Rotundas of any kind; and Domes, entire; or where a part is deficient by means of a Lanthorn, or otherwiſe, at the Top; the Surface between any two parallel Circles being determined by the laſt.

MENSURATION of SOLIDS.

AS, in Menſuration of Superficies, the whole Buſineſs is to find a Rectangle equal to the Figure propoſed; and to determine how many Squares of a certain Dimenſion, the Figure is equal to; ſo, Menſuration of Solids conſiſts in determining how many cubical Feet, &c. are contained in the propoſed Solid.

In order to which, it is neceſſary to know the affinity and proportion between one Solid and another, as contained in the 7th and 8th Books of theſe Elements; and particularly Parallelopipeds, to which all other Solids muſt neceſſarily be reduced, in Menſuration; a right angled Parallelopiped being of the ſame importance, in reſpect of Solids, as the Rectangle amongſt Plane Figures. Alſo, as a Square is the Criterion of ſuperficial meaſure, ſo a Cube, being the moſt perfect Parallelopiped, is the ſtandard, by which the Quantities of Solids are compared and eſtimated.

For, the Definition of a Cube, ſee Def. 10. 7.



1. Let AGE be a Cube, whoſe Sides, AB, &c. are each equal to 3 feet; alſo, let a c d be a Cube, whoſe Side is one foot, or inch, &c.

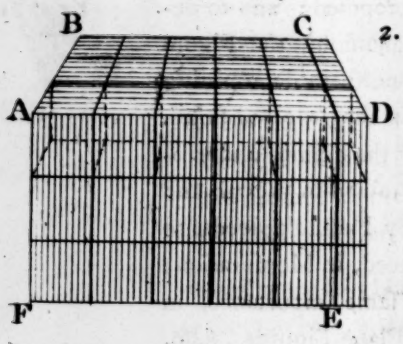
Now, ſuppoſe the Cube a c d to be the Unit of meaſure; it is required to know, how oft the Unit, a c d, is contained in A G E.

It is evident, ſeeing, the Meaſure of AB, BG, &c. are each equal to three times a b, b c, &c. that the Face A B C D, contains the Square b d, 9 times; every other Face, B C F G, or C D E F, the ſame; therefore, each ſmall Cube *a b c*, *d e f*, *e f g*, &c. made by Sections of Planes through *a g*, *g l*, &c. as in the Figure, are equal to a c d. Wherefore, ſince each Surface contains 9 Squares, as BF; and if B *a*, C *g*, &c. be one foot in thickneſs, conſequently, the Solid *a* G *l*, contains the ſmall Cube, a c d, 9 times.

But, AB is equal to three times a b; wherefore, the whole Cube A G E contains the ſmall one 27 times.

For, if it be ſuppoſed to be cut, by parallel Planes, through *a g l* and *b i k*, parallel to the Top and Baſe, the Parts *a* G *l*, *a i l*, and *b* D *k* are equal, ſeeing that their Surfaces are equal; and

and being alſo of equal thickneſs, C*g*, *g i*, *i* D; conſequently, B *g* F, *a il*, and *b* D *k* are equal (15. 7.) But, B *g* F G is equal to 9 times a c b; therefore, AGE is equal to three times 9, = 27.



2. It is manifeſt, that if the Solid A C E (being a Parallelopiped) was longer, equal 5 times a b, having equal thickneſs, it would contain the ſmall Cube, a c b, 45 times, i.e. five times 9; and thus it will increaſe, as often as the meaſure a b is added in length.

Suppoſe half a b be added; it will contain half 9 Cubes, i. e. $4\frac{1}{2}$; ſeeing that the Surface of the End, or Section through C E, contains 9 Squares, and the Solid C E has but half a Cube, i. e. half a b in thickneſs. Therefore, the whole Solid A C E contains the Cube a c b 49 times and a half.

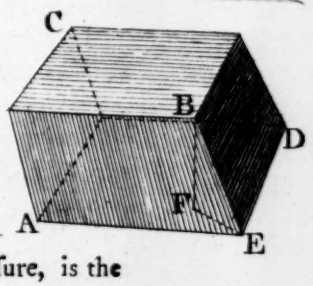
Hence, the meaſure of a right-angled Parallelopiped is to multiply the Side AD by AB, which gives the Area of one Surface (the Top, or Baſe) and, that Product being multiplied by its height or thickneſs, AF, i. e. by the number of times it contains a b (as, in this Caſe, three times) the Product of that multiplication is the Solid Contents of the Parallelopiped A F H.

But, all Parallelopipeds, having equal Baſes and Altitudes, are equal (17. 7.) Therefore, whether it be right or acute angled, having obtained the Area of any one Face, in ſquare meaſure (Art. 12.) that Product being multiplied by the perpendicular Diſtance, between that Face and its oppoſite, gives the ſolid Contents of the Parallelopiped. e. g.

3. Let A B C D be an acute angled Parallelopiped whoſe Baſe A D is 7 Feet, 9 Inches, on one Side, AE; the other, ED, 5 Feet, 4 Inches, its Area in ſquare meaſure, is 41 Feet, 4 Inches.

7-9	
5-4	
38-9	
2-7	
41-4	
3-8	
124-0	
27-6-8	
151-6-8	

The height of the Parallelopiped, BF, being 3 Feet, 8 Inches, it is obvious, that the whole Solid contains 3 times 41-4, and $\frac{2}{3}$; i. e. as often as the meaſure of a Foot is contained in the height, BF, ſo often the Solid contains cubical Feet, as its Baſe contains ſquare Feet.



But, the Perpendicular BF, is 3 Feet and 8 Inches; conſequently, the Solid will contain 41-4, 3 times and two thirds, (8 Inches, being two thirds of a Foot.) Wherefore, 41 Feet, 4 Inches multiplied 3 times, and two thirds, equal 151 Feet, 9 Inches, and 8 Parts, is the true Area, or ſolid Contents of the Parallelopd. ABCD.

Note. What is here meant by an Inch, in ſolid meaſure, is the twelfth part of a Cube Foot; viz. a Foot ſquare, and one Inch in thickneſs; or, that Quantity in any other Figure. And the Part, in Solid meaſure, is 12 cubical Inches; or 12 Inches in length, and one Inch in breadth and thickneſs: a cubical Inch is, conſequently, a Second of a cubical Foot.

It is obvious, that the Side BG, being inclined to the Baſe AD, is longer, than the Perpendicular BF, and conſequently cannot give its true Area, for (joining G F) B F G is a right angled Triangle, of which BG, is the Hypothenuſe, therefore it is longer than B F (12. 1.)

Hence it is manifeſt, that all Parallelopipeds, or Priſms whatever, (for Parallelopipeds are Priſms) having equal height, have that Proportion to each other, which is between their Baſes. And, having equal Baſes, they are conſequently, as their Altitudes.

The Rule, therefore, for meaſuring any Priſm whatever, is to multiply the ſuperficial Area of its Baſe, by its perpendicular height.

And conſequently, the ſame Rule is applicable for Cylinders-5.8.

4. Every Pyramid is equal to the third part of a Priſm having equal Baſes and equal Altitudes. - - - - - Th. 4. 8.

Alſo, every Cone is equal to the third part of a Cylinder - 5.8.

And conſequently, Pyramids and Cones, having equal Baſes and Altitudes, are equal.

Hence the Contents of Pyramids and Cones are obtained; viz. by multiplying the Areas of their Baſes by one third of their height.

Or, if multiplied by the whole height, it gives the Contents of a Priſm or Cylinder, of equal height; conſequently, a Pyramid, or Cone, is one third part of ſuch a Priſm, or Cylinder.

5. To find the ſolid Contents of a Sphere.

Having obtained its Diameter; find the Area of a Circle of that Diameter (Art. 8.) which being multiplied, by the Diameter, gives the Contents of a Cylinder, whoſe height and Diameter are equal.

The Sphere is equal to two thirds of ſuch a Cylinder-Th. 9. 8.

Or, having obtained the Area of a large Circle, of the Diameter of the Sphere; multiply the Product by two thirds of the Diameter.

Otherwiſe. Every Sphere is equal to a Cone, whoſe Baſe is equal to the Surface of the Sphere, and its Altitude to the Radius-17.8.

Therefore, having obtained the Area of its Surface, multiply that Area by one ſixth part of the Diameter.

See an Example, in each, in the Margin.

Let the Diameter of a Sphere be 15; the Circumference of a Circle, of that Diameter, is 47,14, nearly; by the Ratio of 7 to 22.

Then, by Art. 8. the Semicircumference (23, 57) being multiplied by the Radius (7,5, half 15) gives the Area of a great Circle of that Sphere, equal 176,775; which multiplied by 10 (two thirds of the Diameter) gives 1767,75, or three fourths of the Integer.

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23,57
 7,5
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11785
16499
------
176,775
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1767,75
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Multiplying by 10, is only giving one place more of Integers; ſeeing that, a Cypher, being added, makes no difference in the Decimal; 750 thouſandth parts, being equal to 75 hundreth parts; i. e. equal to 3 fourths.

Secondly. 15 being the Diameter of the Sphere, find the Area of its Surface (Art. 21.) which is equal to a Circle whoſe Radius is equal to the Diameter of the Sphere.

The Circumference of every Circle has the ſame Ratio to its Diameter (Cor. 1. 14. 6. El.) conſequently, the Circumference of a Circle, whoſe Area is equal to the Surface of the Sphere, is double the Circumference of the Sphere; half of which, is equal to the whole of the other, equal 47,14.

Multiply

Multiply 47,14 by 15, the Diameter of the Sphere, the Product is the Area of the Surface of the Sphere.

Now, since the Contents of the Sphere is equal to a Cone, the Area of whose Base is 707,1; and its Altitude equal to its Radius, viz. 7,5; and a Cone is equal to one third of a Cylinder of the same Base and Altitude, therefore, 707,1, multiplied by 2,5, one third of 7,5, will produce the same Contents, as before, viz. 1767,75; as in the Margin.

47,14
15,
707,1
2,5
35355
14142
1767,75

6. The solid Contents of a Segment of a Sphere is obtained from the 17th of the 8th, Cor. 3rd.

The solid Contents of the Sphere being equal to a Cone, whose Base is equal to the Surface of the Sphere, and its Altitude to the Radius, as above; and consequently, the Contents of any Sector of a Sphere, being equal to a Cone whose Base is equal to the spherical Surface, and its Altitude to the Radius; (Cor. 1.) it is manifest, that a Segment of the Sphere having the same portion of the Spheres Surface, as a Sector, is equal to the Difference between the Sector and a Right Cone on the Base of the Segment, and, its Side equal to the Radius of the Sphere.

Thus, having shewn how the solid Contents of regular geometrical Solids are obtained, the application of it, in complex ones, will not be difficult; being well versed in the Theory, and having a solid foundation, for the whole, in Geometry.

7. In respect of Guaging; are not almost all large Vessels, either Cylinders or Frustrums of Cones? and consequently they are measured on that Principle; allowing so many Gallons, Quarts, &c. to a cubical Foot, as it is easily known to contain.

Barrels, of all kinds, are measured as two Frustrums, whose common Base is the Diameter at the Bung; but being almost wholly curved, from the Head to the middle, it will be truer to consider it as four, or six Frustrums, two and two, of which, are equal; or the middle part as a Cylinder.

8. The true ſolid Contents of irregular Solids cannot be obtained, otherwiſe than by covering them with Water, or other Fluid, in ſome regular Veſſel; and by taking it out, the Difference, in the fulneſs of the Veſſel, is its Contents.

In meaſuring the Trunks of Trees, &c. which are, generally conical, it is uſual to take the Girt in the middle, in order to reduce them to right angled Parallelopipeds; which, if the intention be to produce the real ſolid Contents, is very erroneus. It is well known, that a Circle contains the greateſt Area of any other plane Figure, having an equal Perimeter; and in Art. 10. is ſhewn the great difference between a Circle and a Square, of equal circuit. Conſequently, a Cylinder would be to a right angled Parallelopiped, of equal length, and two of its oppoſite Faces, (i. e. its Ends) Squares, whoſe Sides are, each, a fourth part of the Circumference of the Baſe, of the Cylinder, as the Circle is to the Square; the difference of which is very conſiderable. (See Art. 10. of Superficies.)

And alſo (being the Fruſtrum of a Cone) if the Girt be taken in the middle, in order to reduce it to a Cylinder, it is alſo erroneous; ſeeing, it is obvious, that the Fruſtrum of the Cone is more than ſuch a Cylinder; the exceſs of the greater End being conſiderably more than the deficiency of the other. But, as I do not intend to treat at large on thoſe matters, and having gone beyond my firſt deſign, I ſhall only obſerve, that, in this Caſe, the Fruſtrum of a Cone is equal to the Difference, between the whole Cone and the leſſer Cone, ſuppoſed to be cut off, from the Vertex.

FINIS, of MENSURATION.

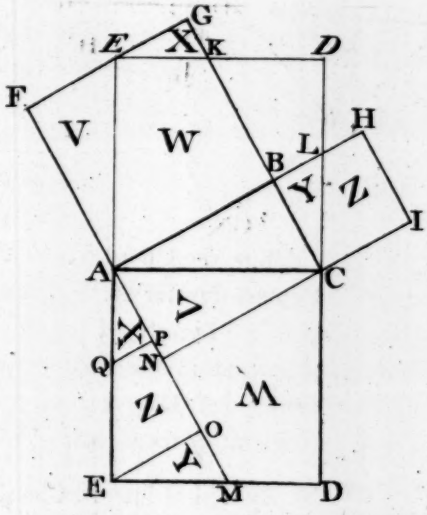
A mechanical and ocular Demonſtration of the 20th Theorem, Book 1ſt.

In Theo. 20. of the firſt Book of Elements, it is demonſtrated, that the Square of the Hypothenuſe, of every right-angled Triangle, is equal to both the Squares of the two Sides, containing the Right Angle. It may be entertaining to ſome, and conviction to others, to give ocular Demonſtration of it; by cuting the two Squares of the Sides, in ſuch wiſe, that they ſhall exactly cover the Square conſtructed on the Hypothenuſe.

On the right-angled Triangle A B C, deſcribe the three Squares, A C D E, A F G B, and B H I C.

Let the Square A E D C, of the Hypothenuſe, be ſuppoſed to be turned over, on AC, in the inverted poſition A *E D* C, cuting the two other Squares; B F, in A*E*, *E* K, and, the Square B I, in C L.

The two Trapeziums, W and Z, and the three Triangles V, X, and Y, will exactly coincide with the ſame Figures, as they are deſcribed on the Square A C D E.



Or, having drawn the Squares as directed, produce E A to *E*, and D C to L; and draw *E*K, perpendicular to A E. Alſo, produce FA to M, and I C to N; draw E O perpendicular to AM, make O P equal B C, (equal EO) and draw P Q parallel to EO.

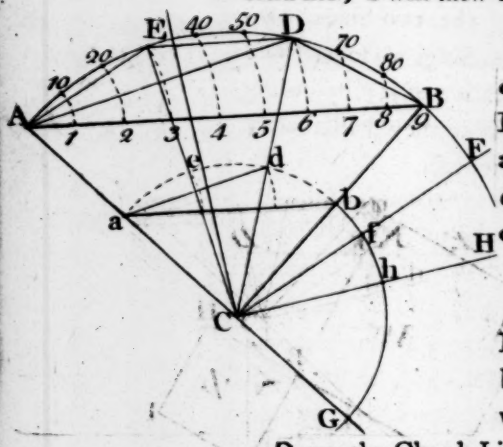
The Lines being drawn on the three Squares, as they are deſcribed, in the Figure annexed, on ſtrong Paper or Paſteboard, and the two Squares, F B and B I, being cut, carefully, with a fine thin edged knife, or otherwiſe, into five pieces, as in the Figure, they will, each, coincide with the correſponding Figures in the Square A C D E,

If the poſition of the Letters, V, W, X, &c. be regarded, there needs no other directions in reſpect of placing them.

The

The Line of Chords, and its Uſe, explained.

As the Line of Chords, and its Uſe, is better known than underſtood, I preſume, a Deſcription of it may not be unneceſſary. And firſt, I will ſhew how to conſtruct a Line of Chords.



With any Radius, at Diſcretion, as AC, on C deſcribe the Ark A D B, of 90 Degrees; i. e. make ACB a Right Angle; and on the Center, C, with any Radius, AC, draw ADB; which will be a fourth part of the circumference of a Circle.

Divide the Ark ADB into 9 equal Parts. Firſt, make AD equal AC and divide DB into three equal parts, and AD will be equal ſix of thoſe diviſions, each of which will be 10 Degrees of that Circle.

Draw the Chord Line AB; and, from the extreme A, as a Center, transfer all the Meaſures on the Ark, to the Chord AB, as in the Figure. Each Diviſion being ſubdivided, into ten, and transfered in the ſame manner, AB will then be a Scale for any number of Degrees, from one to ninety; which gives a Right Angle at C, the Center of the Circle, of which, AC is the Radius.

N. B. This Line of Chords is upon every common Scale, in a Caſe of Inſtruments, for drawing; the Uſe, of which, is the ſame as the Protractor, deſcribed in the Theory of Plane Angles; viz. to meaſure any Plane Angle; or, to lie down any Angle required, on a Plane, the meaſure of the Angle, that is, the number of Degrees it contains, being known.

In applying the Line of Chords to uſe, take 60 Deg. from A, on the Line AB, with your Compaſſes; and, with that Radius, on any Point, C, of the Line AC (at which Point an Angle is required, with that Line, of ſome known meaſure) deſcribe the Ark AEB, cuting AC in A; from which Point, ſet off the number of Degrees, on the Ark, which is the Meaſure of the Angle required.

e. g. Suppoſe an Angle of 35 Degrees is wanted, take 35 Deg. from the Point A on the Line of Chords, and ſet it off from A to E, draw EC; then, ACE is an Angle of 35 Deg.

If an Angle of 60 Degrees was required; make AD equal to the Radius, and draw DC; then is ACD an Angle of 60 Degrees. For ADC is an Equilateral Triangle, whoſe Angles are all equal, and conſequently, the Chord, AD, is equal to the Radius AC-11.4.

If a Right Angle be required; from A make AB, equal to the whole Line of Chords of 90 Degrees, and draw BC, which will be perpendicular to AC.

Thus, may any Angle, leſs than 90 Degrees be readily obtained. But, if an obtuſe Angle is wanted, as 105 Degrees, with the Line AC, at the Point C; having firſt ſet of 90 Degrees, from A to B, or 60, to D; take the remainder BF, 15 Degrees or DF 45, and draw FC, making the Angle ACF, 105 Degrees. Thus may any Angle whatever be deſcribed.

Or, an obtuſe Angle may be thus taken, at once, if there be room on your Paper.

If an Angle ACF, of 105, or ACH, 125, be required; ſubtract the numberof Degrees from 180, the Sum of two Right Angles, the Difference is 75, or 55.

Produce AC towards G; with the Radius AC, equal AD, (i. e. to 60 Degrees, on the Line of Chords,) deſcribe the Ark GHF, and ſet off GF, 75, or GH 55 Degrees, and draw CF, or CH, making the Angle required.

After the ſame manner an Angle may be meaſured; by deſcribing an Ark, on its Vertex, with the Radius of 60 Degrees, cuting both Sides; as AE, cuting AC and CE, of the Angle ACE; then, the meaſure of the Chord Line, AE (applied to AB) ſhews the number of Degrees that Angle contains.

The Reaſon of all which is ſo very obvious, it needs no further explanation; but, be ſure that you always take your meaſure, or number of Degrees on the Line of Chords, begining at the Point A; becauſe it is evident, that they are continually diminiſhing, from A to B, although they are equal on the Ark; for the Diviſions on AB are the Chords of each Ark, ſet off from A towards B; each of which, is ſtill leſs in proportion to the Ark.

The Chord of 10 Degrees, it is evident, deviates very little from a Right Line; the Chord of 30 Degrees deviates conſiderably, and AD, the Chord of 60, ſtill more; for the Chord AD, of 60, and the two Chords of 30 make a Triangle, AED; of which, it is manifeſt, that, the two Sides AE, ED, each of 30, Degrees is greater than AD, the Chord of 50. (13. 1. El.)

If it ſhould be aſked, why 60 Degrees, particularly, is taken for Radius, when we begin the Operation of taking or deſcribing an Angle, by the Line of Chords? the Reaſon is plain; becauſe the Chord of 60 Degrees is always equal to the Radius; and, the Side, AD, of the Equilateral Triangle ADC, is the Side of a regular Hexagon inſcribed in a Circle, of that Radius; and conſequently, it is equal to a ſixth Part of the Circumference, or 60 Degrees. Hence it is clear, that no other Radius can anſwer to that Line of Chords.

FINIS.

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